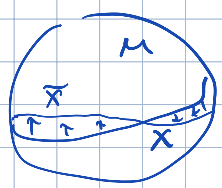
Correction to last time:

If $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ oriented v.bun of odd rank k, then $\boxed{e(E) + e(E) = 0}$ (Euler class)

rather than $e(E) = 0$

(i.e. $e(E)$ is a 2-torsion elt in $H^k(M;\mathbb{Z})$)

• Typical application of the Euler class: $X \hookrightarrow M$ submanifold.



Can X be displaced infinitesimally, $X \to \tilde{X}$ inside M so that $X \cap \tilde{X} = \emptyset$?

- This is measured by the Euler class of the normal bundle,

$$e(NX) \in H^{\dim M - \dim X}(X;\mathbb{Z})$$

for $\dim M = 2\dim X$, $\langle e(NX), [X] \rangle =$ "self-intersection number" (of $X \subset M$)

Construction of Chern classes

For $\begin{matrix} E \\ \downarrow \pi \\ M \end{matrix}$ a rk $= k$ cx. v.bun., choose a Herm. metric $\langle , \rangle$.

$$\begin{matrix} SE = \{(x, v \in E_x) \mid \langle v,v \rangle = 1\} \\ \downarrow \pi \\ M \end{matrix}$$ - unit sphere bundle

Let $\begin{matrix} \check{E} \\ \downarrow \\ SE \end{matrix}$ be the rk $= k-1$ cx. v.bun over SE with fiber $(\check{E})_{x,v} = \{u \in E_x \mid \langle u,v \rangle = 0\}$

$= (\mathrm{Span}_{\mathbb{C}}\{v\})^{\perp} \subset E_x$

Fact: for $i < 2k-1$, $\pi^*: H^i(M) \to H^i(SE)$ is an isomorphism

(follows from Gysin sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H^{i-2k}(M) \arrow[r, "\cup e"] & H^i(M) \arrow[r, "\pi^*"] & H^i(SE) \arrow[r] & H^{i-2k+1}(M) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^i(E,E_0) \arrow[r] \arrow[u, "\wr", "\text{Thom iso}"'] & H^i(E) \arrow[r] \arrow[u, equal] & H^i(E_0) \arrow[r] \arrow[u, equal] & H^{i+1}(E,E_0) \arrow[r] \arrow[u, "\wr"] & \cdots
\end{tikzcd}$$

$H^{i-2k}(M) = 0$ for $i < 2k-1$, $H^{i-2k+1}(M) = 0$)

def Chern classes $c_i(E) \in H^{2i}(M;\mathbb{Z})$ are defined by induction in $\mathrm{rk}(E)$:

- Top Chern class $\boxed{c_k(E) := e(E_{\mathbb{R}})}$ - the Euler class.
- For $i < k$, $\boxed{c_i(E) := (\pi^*)^{-1} \underbrace{c_i(\check{E})}_{\in}}$

$$H^{2i}(M) \longleftarrow H^{2i}(SE)$$

- For $i > k$, $c_i(E) = 0$.

Total Chern class: $c(E) := 1 + c_1(E) + \dots + c_k(E) \quad \in H^*(M;\mathbb{Z})$

it has an inverse $c(E)^{-1} = 1 - c_1(E) + (c_1(E)^2 - c_2(E)) + \dots$

Properties

- Naturality: if
$$\begin{tikzcd} E \arrow[r, "\Phi"] \arrow[d] & E' \arrow[d] \\ M \arrow[r, "f"] & M' \end{tikzcd}$$
a map of cx. vector bundles inducing linear iso. in fibers,

then $c(E) = f^* c(E')$

- Stability: $c(E \oplus \underline{\mathbb{C}}^k) = c(E)$

- Product rule: $c(E \oplus E') = c(E)\, c(E')$

- If $\begin{array}{c}\overline{E} \\ \downarrow \\ M\end{array}$ - conjugate bundle (same underlying $E_{\mathbb{R}}$ but the opposite complex structure $-J$)

Then $c_i(\overline{E}) = (-1)^i c(E)$.

Note: given a Herm. metric, one has an iso

$$\overline{E} \simeq E^* = \mathrm{Hom}_{\mathbb{C}}(E, \underline{\mathbb{C}})$$

as in real case

$$\{\text{rk } k \text{ cx. v. bun over } M\}/_{iso} \xleftrightarrow{1-1} [M, Gr_{\underline{\mathbb{C}}}(k,\infty)]$$

$$\begin{array}{c} E \\ \downarrow \\ M \end{array} \longmapsto f_E : M \to Gr \quad \text{classifying map}$$

$$E = f^* \tau^k_{\mathbb{C}} \longleftarrow\!\!\shortmid \; f: M \to Gr$$

$\tau^k_{\mathbb{C}}$: cx. taut. bundle over $Gr_{\mathbb{C}}(k,\infty)$

Thm $H^\bullet(Gr_{\mathbb{C}}(k,\infty), \mathbb{Z}) = \mathbb{Z}[c_1, c_2, \dots, c_k]$ - polynomial ring generated by Chern classes $c_1(\tau^k), \dots, c_k(\tau^k)$. There are no relations between these generators.

Rem: $Gr_{\mathbb{C}}(k,m)$ has a Schubert cell decomposition with only $\underline{\underline{\text{even-dimensional}}}$ cells (corresp. to echelon forms) $\Rightarrow$ all coboundary maps are zero.

Ex: $H^\bullet(\mathbb{C}P^\infty; \mathbb{Z}) = \mathbb{Z}[c_1]$

$c_1 = c_1(\tau^1)$ — generator in $H^2(\mathbb{C}P^\infty)$

Also: $H^\bullet(\mathbb{C}P^n; \mathbb{Z}) = \mathbb{Z}[c_1] / c_1^{n+1} = 0$

Ex: For $T\mathbb{C}P^n$ - the tangent bundle of $\mathbb{C}P^n$, seen as a complex rk = n v. bun,

$$\boxed{c(T\mathbb{C}P^n) = (1+a)^{n+1}} \in H^\bullet(\mathbb{C}P^n, \mathbb{Z})$$

total Chern class, where $a = -c_1(\tau^1_{\mathbb{C}})$ - generator of H^2 = Poincaré dual class of the (2n-2)-cycle $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$

⌈ - as in $\mathbb{R}P^n$ case: $T\mathbb{C}P^n = Hom(\tau, \tau^\perp)$

$$Hom(\tau, \tau^\perp \oplus \tau) = Hom(\tau, \tau^\perp) \oplus \underline{\mathbb{C}}$$

$$\underbrace{\tau^* \oplus \dots \oplus \tau^*}_{n+1} \qquad \Rightarrow \quad c(T\mathbb{C}P^n) = c(\tau^*)^{n+1} = (1+a)^{n+1}$$ ⌋

[note: for a real v.bun., $E^* \underset{iso}{\simeq} E$ by a choice of metric;
for a cx v.bun., $E^* \simeq \bar{E} \not\simeq E$ - not iso! (Herm. metric)
Ex: $\tau^* \not\simeq \tau$ for $\mathbb{C}P^n$ ($c = 1+a$, $c = 1-a$)]

Thus, $c_i(\mathbb{C}P^n) = \binom{n+1}{i} a^i$, $c_n(\mathbb{C}P^n) = (n+1)\, a^n$ = Euler class $e(T\mathbb{C}P^n)$,

$\langle \;\text{———}\;, [\mathbb{C}P^n] \rangle = n+1$ = Euler char. of $\mathbb{C}P^n$

Remark (on c_1) for cx v. bun. $/M$, $c_1(E \otimes E') = c_1(E) + c_1(E')$
$c_1(\underbrace{E^*}_{\cong \bar{E}}) = -c_1(E)$

specializing to line bundles:

$$f: \{\text{cx line bundles over } M\}/_{\text{iso}} \longrightarrow H^2(M, \mathbb{Z})$$
$$\begin{matrix} E \\ \downarrow \\ M \end{matrix} \longmapsto c_1(E)$$

is a) a bijection
b) a group homomorphism w.r.t. $\otimes$ for line bundles and addition in H^2

$$\left(\{\text{cx. line bundles}/M\}/_{\text{iso}} \underset{\text{classifying map}}{\overset{1-1}{\longleftrightarrow}} [M, \underbrace{\mathbb{C}P^\infty}_{K(\mathbb{Z},2)}] \simeq H^2(M, \mathbb{Z}) \right)$$

c_1

$K(\mathbb{Z},2)$ - Eilenberg-Maclane space

Pontrjagin classes

For $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ a real $\overset{\text{rk}=k}{\text{v.b.}}$, one has a complexification $\begin{matrix} E \otimes \mathbb{C} \\ \downarrow \\ M \end{matrix}$ - a cx. rk $=k$ v.b.,
with $(E \otimes \mathbb{C})_x = (E_x) \otimes \mathbb{C}$
= complexification of fibers

- $E \otimes \mathbb{C} \cong E \oplus E$ - can. iso
$(x, \xi_1 + i\xi_2) \mapsto (x, \xi_1, \xi_2)$
$$\begin{tikzcd} E \otimes \mathbb{C} \arrow[rr, "\cong"] \arrow[dr] & & E \oplus E \arrow[dl] \\ & M & \end{tikzcd}$$

- $E \otimes \mathbb{C}$ is iso. to the conjugate $\overline{E \otimes \mathbb{C}}$
$(x, \xi_1 + i\xi_2) \longmapsto (x, \xi_1 - i\xi_2)$

- $c(E \otimes \mathbb{C}) = 1 + c_1(E \otimes \mathbb{C}) + c_2(E \otimes \mathbb{C}) + \cdots + c_k(E \otimes \mathbb{C})$
$c(\overline{E \otimes \mathbb{C}}) = 1 - c_1(E \otimes \mathbb{C}) + c_2(E \otimes \mathbb{C}) + \cdots \pm c_k(E \otimes \mathbb{C})$

$\Rightarrow$ for i odd, $c_i(E \otimes \mathbb{C})$ are elements of order 2.

def: i-th Pontrjagin class $p_i(E) \in H^{4i}(M; \mathbb{Z})$ is defined as
$$p_i(E) := \underset{\substack{\uparrow \\ \text{conventional sign}}}{(-1)^i} c_{2i}(E \otimes \mathbb{C})$$
(here we are ignoring c_{odd} - the 2-torsion classes)

- $p_i(E) = 0$ for $i > \frac{k}{2}$
- $p(E) = 1 + p_1(E) + \ldots + p_{[\frac{k}{2}]}(E) \in H^\bullet(M; \mathbb{Z})$ - total Pontrjagin class

- P. classes are natural w.r.t. bundle maps (which are iso in fibers);
- $p(E\oplus \underline{\mathbb{R}}^{\ell}) = p(E)$
- $p(E\oplus E') = p(E)\,p(E')$ modulo elts of order 2. (from multiplicativity of Chern classes)

i.e., $2\,p(E\oplus E') = 2\,p(E)\,p(E')$

Ex: $TS^n \oplus \overbrace{NS^n}^{\cong\underline{\mathbb{R}}} = \begin{matrix}\underline{\mathbb{R}}^{n+1}\\ \downarrow \\ S^n\end{matrix}$ - triv. bundle $\Rightarrow p(TS^n) = 1$

NS^n: normal for $S^n \hookrightarrow \mathbb{R}^{n+1}$

$$\begin{tikzcd}
& \text{real v.bun} \arrow[dr, "\otimes\mathbb{C}"] & \\
\text{oriented v.bun} \arrow[ur] & & \text{cx v.bun} \arrow[ll, "(\ )_{\mathbb{R}}"]
\end{tikzcd}$$

going around:

- E (real v.b.) $\longrightarrow (E\otimes\mathbb{C})_{\mathbb{R}} \overset{\text{can iso}}{\sim} E\oplus E$
- ω (cx. v.b.) $\longrightarrow \omega_{\mathbb{R}}\otimes\mathbb{C} \overset{\text{can iso}}{\sim} \omega\oplus\bar{\omega}$ Lm 15.4 in M-S

$a+ib\in\omega_x$
$\updownarrow$
$(a,b)\in(\omega_{\mathbb{R}})_x$
$\updownarrow$
$(a,b)\in(\omega_{\mathbb{R}}\otimes\mathbb{C})_x$
$\updownarrow$
$\left(\frac{a+ib}{2}, \frac{a-ib}{2}\right)\in(\omega\oplus\bar{\omega})_x$

Lemma For $\begin{matrix}\omega\\ \downarrow\\ M\end{matrix}$ a cx. rank k v.bun., Chern classes $c_i(\omega)$ determine Pontryagin classes $p_j(\omega_{\mathbb{R}})$ by

$$1-p_1+p_2-\dots\pm p_k = (1-c_1+c_2-\dots\pm c_k)(1+c_1+c_2+\dots+c_k)$$

Ex: ω = tangent bundle of $\mathbb{CP}^n$

$$1-p_1+p_2-\dots\pm p_n = \underbrace{(1-c_1+c_2-\dots\pm c_n)}_{(1-a)^{n+1}}\underbrace{(1+c_1+\dots+c_n)}_{\substack{(1+a)^{n+1}\\ \text{Chern polynomial}}} = (1-a^2)^{n+1}$$

$$\Rightarrow \boxed{p_i(T\mathbb{CP}^n) = \binom{n+1}{i}a^{2i}} \qquad i=1,\dots,\left[\tfrac{n}{2}\right]$$

n	$p(\mathbb{CP}^n)$
1	1
2	$1+3a^2$
3	$1+4a^2$
4	$1+5a^2+10a^4$

p_1 p_2