# LAST TIME

**Thm:** Let $\begin{matrix} E \circlearrowleft G \\ \downarrow \\ B \end{matrix}$ be a G-bundle with E aspherical. Then this bundle is universal:

$\forall$ top space M of the homotopy type of a CW complex, the map

$$\begin{tikzcd} \Phi : [M, B] \arrow[r] & \{G\text{-bundles over } M\}/_{\text{iso}} \\ f \arrow[r, mapsto] & \begin{matrix} f^*E \\ \downarrow \\ M \end{matrix} \end{tikzcd}$$

is a bijection.

$$\Phi^{-1}\left(\begin{matrix} P \circlearrowleft G \\ \downarrow \\ M \end{matrix}\right) =: \left(\text{classifying map } f_P : M \to B\right)$$

**Cor:** If $\begin{matrix} E \circlearrowleft G \\ \downarrow \\ B \end{matrix}$, $\begin{matrix} E' \circlearrowleft G \\ \downarrow \\ B' \end{matrix}$ two univ. G-bundles, then there exists a G-bundle map

$$\begin{tikzcd} E \arrow[r, "\tilde{h}"'] \arrow[d] & E' \arrow[d] \\ B \arrow[r, "h"'] & B' \end{tikzcd}$$

with h a homotopy equivalence.

**Notation**

$$\begin{matrix} EG := E \\ \downarrow \\ BG := B \end{matrix}$$

– univ. G-bundle

$BG := B$ ← well-defined up to homotopy

- $\pi_{n-1}(G) = \pi_n(BG)$

**Ex:** For $G$ a discrete group, $BG = K(G,1)$ - Eilenberg-MacLane space

(recall: $X = K(G,n)$ if $\pi_i(X) = \begin{cases} G, & i=n \\ 0, & i\neq n \end{cases}$ )

univ. bundle:

$$\begin{tikzcd} EG = \overbrace{K(G,1)}^{\text{univ. cover}} \circlearrowleft G \arrow[d] \\ BG = K(G,1) \end{tikzcd}$$

**Ex:** $G = \mathbb{Z} \rightsquigarrow BG = S^1$,

$$\begin{tikzcd} EG \arrow[d] & = & \mathbb{R} \circlearrowleft \underset{x \mapsto x+k}{\mathbb{Z}} \arrow[d] \\ BG & & S^1 = \mathbb{R}/\mathbb{Z} \end{tikzcd}$$

antipodal map

**Ex:** $G = \mathbb{Z}_2 \longrightarrow$

$$\begin{tikzcd} EG = S^\infty \circlearrowleft \mathbb{Z}_2 = \varinjlim_n S^n \arrow[d] \\ BG = \mathbb{RP}^\infty = S^\infty/\mathbb{Z}_2 \end{tikzcd}$$

(under $S^1 \subset S^2 \subset S^3 \subset \dots$ )

contractible!

**Ex:** $G = U(1) = S^1$ (not discrete!)

$$\begin{tikzcd} EG = S^\infty = \varinjlim S^{2n-1} \circlearrowleft S^1 \arrow[d] \\ BG = \mathbb{CP}^\infty \end{tikzcd}$$

$S^{2n-1} \subset \mathbb{C}^n$

n-tuple of

$\{(v_1, \dots, v_n)$ - lin. indep. vectors $\in \mathbb{C}^N\}$

$\parallel$ - Stiefel manifold

**Ex:** $G = GL(n,\mathbb{C})$

$$\begin{tikzcd} EG = V_{\mathbb{C}}(n,\infty) \circlearrowleft GL(n,\mathbb{C}) = \varinjlim_N V_{\mathbb{C}}(n,N) \arrow[d] \\ BG = Gr_{\mathbb{C}}(n,\infty) \end{tikzcd}$$

contractible! (Cohen, p 85)

**Ex:** $G = U(n)$
unitary group

$$\begin{tikzcd} EG = V^{U}(n,\infty) = \lim\limits_{N} V^{U}(n,N) \arrow[d] \arrow[loop above, "U(n)"] \\ BG = Gr_{\mathbb{C}}(n,\infty) \end{tikzcd}$$

$V^U(n,N) = \{(v_1,\dots,v_n) \in (\mathbb{C}^N)^{\times n} \mid \langle v_i, v_j\rangle = \delta_{ij}\}$
- unitary Stiefel mfd

**Rem:** $BGL(n,\mathbb{C}) = BU(n)$ corresponds to the fact that each cx. v. bun. has a unique up to homotopy Herm. metric ( $U(n)$-structure )

similarly,

**Ex:** over $\mathbb{R}$:

$$\begin{tikzcd} EGL(n,\mathbb{R}) = V_{\mathbb{R}}(n,\infty) \arrow[dr] \arrow[loop above, "GL(n{,}\mathbb{R})"] & & V^{O}(n,\infty) = EO(n) \arrow[dl] \arrow[loop above, "O(n)"] \\ & BGL(n,\mathbb{R}) = Gr_{\mathbb{R}}(n,\infty) = BO(n) & \end{tikzcd}$$

- If $H \subset G$ a subgroup, then $EG \circlearrowleft H$ - free action, so

$$\begin{tikzcd} EG \arrow[loop right, "H"] \arrow[d] \\ EG/_{H} =: BH \end{tikzcd}$$

. So, e.g., for $H \subset GL(n,\mathbb{C})$, $BH = V_{\mathbb{C}}(n,\infty)/_{H}$

- Let $\begin{tikzcd} EG \arrow[d] \\ BG \end{tikzcd}$ be the univ. bundle and $H \subset G$ a subgroup, then there is a fiber bundle $\begin{tikzcd} BH \arrow[d] \\ BG \end{tikzcd}$ with fiber the orbit space $G/_{H}$.

( this bundle is:

$$\begin{tikzcd} G/_H \arrow[r] & EG \times_G G/_H = EG/_H = BH \arrow[d] \\ & EG/_G = BG \end{tikzcd}$$

)

**Ex:** $G = O(n)$
$\cup$
$H = SO(n)$

$$\longrightarrow \begin{tikzcd} BH = \widetilde{Gr}(n,\infty) \arrow[d, "2:1"] \\ BO = Gr_{\mathbb{R}}(n,\infty) \end{tikzcd}$$

$\leftarrow$ oriented Grassmanian

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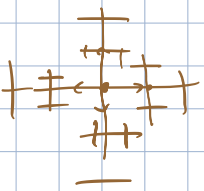
More examples: • $BF_2 = S^1 \vee S^1$ , $EF_2 = \widetilde{S^1 \vee S^1}$ = infinite fractal tree

$\swarrow$ free group on 2 elements



braids on $n$ strands

- $B(Br_n) = \{$ n-element subsets in $\mathbb{R}^2\}$
- $B(PBr_n) = \{$ordered n-elt. subsets in $\mathbb{R}^2\}$
  $\nwarrow$ pure
- $BS_n = \{$ n-elt. subsets in $\mathbb{R}^\infty\}$ , $ES_n = \{$ordered n-elt subsets in $\mathbb{R}^\infty\}$
  $\nwarrow$ sym group

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- $B(G_1 \times G_2) = BG_1 \times BG_2$

## Milnor's join construction for BG <existence of univ. bundles>

- for $X, Y$ top. spaces, join is $X * Y := X \times [0,1] \times Y \Big/ \begin{array}{l}(x,0,y) \sim (x,0,y') \ \forall x \in X; y, y' \in Y \\ (x,1,y) \sim (x',1,y) \ \forall x, x' \in X; y \in Y\end{array}$

Ex: $X * \{y\} = \mathrm{cone}(X)$

point

- $X * \{y_1, y_2\} = \Sigma X$ – suspension
- $\{x_0\} * \{x_1\} * \dots * \{x_n\}$ = n-simplex with vertices at $x_0, \dots, x_n$
- $S^k * S^\ell \simeq S^{k+\ell+1}$

- One has inclusions $X \hookrightarrow X * Y \hookleftarrow Y$. These inclusions are null-homotopic (homotopic to constant maps)

$x \longmapsto [(x,0,y)]$ $\forall y \in Y$

- For $G$ a group, consider the space $G^{*(k+1)} = \underbrace{G * G * \dots * G}_{k+1}$ – it has a free $G$-action

$$(g_0, t_1, g_1, \dots, t_k, g_k) \cdot g = (g_0 g, t_1, g_1 g, \dots, t_k, g_k g)$$

Milnor's construction:

$$\begin{tikzcd} EG := J(G) \circlearrowleft G := \varinjlim_{k} G^{*(k+1)} \arrow[d] \\ BG := J(G)/G \end{tikzcd}$$

limit over inclusions $G^{*(k+1)} \hookrightarrow G^{*(k+2)}$

- $J(G)$ carries a free $G$-action and is aspherical since $S^n \to G^{*(k+1)} \hookrightarrow G^{*(k+2)}$ is homotopic to a constant map, by

Ex: a) $G = \mathbb{Z}_2 \simeq S^0 \Rightarrow G^{*(k+1)} \simeq S^k \circlearrowleft S^0 \Rightarrow J(G) \simeq S^\infty \circlearrowleft S^0$

$$\begin{tikzcd} J(G) \simeq S^\infty \circlearrowleft S^0 \arrow[d] \\ J(G)/G = \mathbb{RP}^\infty \end{tikzcd}$$

b) $G = S^1 \Rightarrow G^{*(k+1)} \simeq S^{2k+3} \circlearrowleft S^1 \Rightarrow J(G) \simeq S^\infty \circlearrowleft S^1$

$$\begin{tikzcd} J(G) \simeq S^\infty \circlearrowleft S^1 \arrow[d] \\ J(G)/G \simeq \mathbb{CP}^\infty \end{tikzcd}$$

Rem

- one has a G-equivar. map $\underset{\substack{\uparrow\\ \text{simplex}}}{\Delta^k} \times G^{k+1} \circlearrowleft_{\text{diag}} G \xrightarrow{\gamma_k} G^{*(k+1)} \circlearrowleft G$

  $\gamma_k$ is a homeo on the interior of $\Delta^k$

$\rightsquigarrow$ $G^{*(k+1)} = \bigcup_k \Delta^k \times G^{k+1} / \sim$ has the structure of a "G-CW complex"

def. [4.1] in Cohen:

$$X = \bigcup_i \underbrace{D^{d(i)} \times G/\overset{\text{subgroup}}{H_i}}_{\text{"disk orbits"}} \Big/ \underset{\substack{\uparrow\\ \text{attaching maps}\\ (G\text{-equiv})}}{\sim}$$

$\rightsquigarrow$ $G^{*(k+1)}/G$ is a CW complex.

## Group cohomology:

$$H^\cdot_{\text{group}}(G) := \underset{\substack{\uparrow\\ \text{(singular) cohomology}}}{H^\cdot}(BG)$$

- group cochain complex: $G$ - (discrete) group
  $A$ - $G$-module

$$C^n(G,A) := \{\text{maps } \varphi : G^{\times n} \to A\} \qquad (*)$$

coboundary map:

$$(d\varphi)(g_1,\dots,g_{n+1}) = \overset{\text{module map}}{g_1}\varphi(g_2,\dots,g_{n+1}) + \sum_{i=1}^{n} (-1)^i \varphi(g_1,\dots,g_i g_{i+1},\dots,g_{n+1}) + (-1)^{n+1}\varphi(g_1,\dots,g_n)$$

$$H^n(G,A) = \frac{\{\varphi \in C^n(G,A) \mid d\varphi = 0\}}{dC^{n-1}(G,A)} \begin{array}{l} \leftarrow \text{group cocycles} \\ \leftarrow \text{group coboundaries} \end{array}$$

- $H^0(G,A) = \underset{\substack{\uparrow\\ G\text{-invariant vectors}}}{A^G} = \{\varphi \in A \mid g\cdot\varphi - \varphi = 0\}$

Ex:

$H^1(\mathbb{Z},\mathbb{Z})$:

$\varphi(k) = ak$

$(d\varphi)(k,l) = \varphi(l) - \varphi(k+l) + \varphi(k) = 0 \quad \forall a \in \mathbb{Z}$

$\to H^1(\mathbb{Z},\mathbb{Z}) \simeq \mathbb{Z}.$

$(\varphi : k \mapsto ak) \leftrightarrow a$

- for $A$ an ab. group with trivial $G$-action,

$$\underset{\substack{\uparrow\\ \text{from the cochain complex } (*)}}{H^n(G,A)} = \underset{\substack{\uparrow\\ \text{cohomology of } BG = \mathcal{J}(G)/G \text{ as a CW complex}}}{H^\cdot}(BG; \underset{\text{coefficients}}{A})$$