Stiefel-Whitney classes.

(ref: Milnor-Stasheff)

context: real v.bun. in Top

Axioms

① To each vector bundle $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ of rank = k, there corresponds a sequence of cohomology classes $w_i(E) \in H^i(M, \underbrace{\mathbb{Z}_2}_{\{0,1\}})$, $i = 0, 1, 2, \ldots$ — the "Stiefel-Whitney classes" of E

singular cohomology

with $w_0(E) = 1 \in H^0(M, \mathbb{Z}_2)$

$w_{>k}(E) = 0$

② Naturality If
$$\begin{matrix} E & \xrightarrow{\Phi} & E' \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & M' \end{matrix}$$
a bundle map inducing isomorphisms on fibers,

then $\boxed{w_i(E) = f^* w_i(E')}$

Cor If E and E' are isomorphic then $w_i(E) = w_i(E')$.

Cor For $E \simeq \underline{\mathbb{R}^k}$ a trivial v. bun., $w_{>0}(E) = 0$

(Since E is iso to pullback of $\begin{matrix} \mathbb{R}^k \\ \downarrow \\ pt \end{matrix}$)

③ Whitney product axiom: Given v. bundles $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$, $\begin{matrix} E' \\ \downarrow \\ M \end{matrix}$, one has

$$w_m(E \oplus E') = \sum_{i=0}^{m} w_i(E) \cup w_{m-i}(E')$$

E.g. $w_1(E \oplus E') = w_1(E) + w_1(E')$

$w_2(E \oplus E') = w_2(E) + w_1(E) w_1(E') + w_2(E')$

. . . .

Cor $w_i(E \oplus \underline{\mathbb{R}}^\ell) = w_i(E)$ for any E

triv bun

Cor if E possesses a nowhere vanishing section, then $w_k(E) = 0$. (k = rk E)

if E has ℓ everywhere lin. indep. sections, then $w_k(E) = w_{k-1}(E) = \ldots = w_{k-\ell+1}(E) = 0$

(Since E then splits as $E' \oplus \underline{\mathbb{R}}^\ell$)

④ For $\begin{matrix} \tau \\ \downarrow \\ \mathbb{R}P^1 \end{matrix}$ the tautological bundle, $w_1(\tau) \neq 0$.

Total Stiefel-Whitney class:

$w(E) := 1 + w_1(E) + w_2(E) + \dots$ $\in \underset{\substack{\uparrow \text{com.} \\ \text{graded algebra } /\mathbb{Z}_2}}{H^\bullet(M, \mathbb{Z}_2)}$ - formal sums of cohom. classes

③ $\Leftrightarrow$ $w(E \oplus E') = w(E) \cdot w(E')$

Rem elements of form $w = \underline{\underline{1}} + w_1 + w_2 + \dots \in H^\bullet(M, \mathbb{Z}_2)$ form a commutative group under multiplication.

In particular, there is an inverse $w^{-1} = 1 + \bar{w}_1 + \bar{w}_2 + \dots$

- Can find $\bar{w}_i$ inductively from $\bar{w}_m = w_1 \bar{w}_{m-1} + w_2 \bar{w}_{m-2} + \dots + w_{m-1}\bar{w}_1 + w_m$

$\Rightarrow \bar{w}_1 = w_1$, $\bar{w}_2 = w_1^2 + w_2$, $\bar{w}_3 = w_1^3 + w_3$, $\bar{w}_4 = w_1^4 + w_1^2 w_2 + w_2^2 + w_4$, ...

Another way: $w^{-1} = (1 + (w_1 + w_2 + \dots))^{-1} = 1 - (w_1 + w_2 + \dots) + (w_1 + w_2 + \dots)^2 - \dots$

$= 1 \underbrace{- w_1}_{\bar{w}_1} + \underbrace{(-w_2 + w_1^2)}_{\bar{w}_2} + \dots$ (can ignore $\pm$)

Cor (of ③) if $E \oplus E'$ is a trivial bundle, then

$$w(E') = w(E)^{-1}$$

Ex: "Whitney duality thm": if $M \overset{i}{\hookrightarrow} \mathbb{R}^m$ immersion, then $TM \oplus \underset{\substack{\uparrow \\ \text{normal bund.}}}{NM} = i^* T\mathbb{R}^m$ - trivial bundle

$\Rightarrow w(TM) = w(NM)^{-1}$ $(*)$

Ex: for $\underset{\text{unit sphere}}{S^n} \subset \mathbb{R}^{n+1}$, $(*)$ implies $w(TS^n) = w(\underbrace{NS^n}_{\text{trivial}})^{-1} = 1$.

So, for TS^n, one has $w_j = 0$, $j > 0$.
(I.e. TS^n cannot be distinguished from the triv. bundle by S-W classes)

Lemma a) $H^i(\mathbb{R}P^n, \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & 0 \le i \le n \\ 0, & i > n \end{cases}$

b) denote a the generator of $H^1(\mathbb{R}P^n, \mathbb{Z}_2)$.
Then $H^i(\mathbb{R}P^n, \mathbb{Z}_2)$ is generated by $a^i = \underbrace{a \cup a \cup \dots \cup a}_{i}$

I.e. $H^\bullet(\mathbb{R}P^n; \mathbb{Z}_2) = \mathbb{Z}_2[a] / {a^{n+1} = 0}$

unital algebra over $\mathbb{Z}_2$ generated by a with relation $a^{n+1} = 0$

<u>Ex</u>: for $\mathbb{RP}^1$, $\underset{\substack{\uparrow \\ \text{tauto line bundle}}}{w(\tau)} = 1 + a$ ← from ②   no higher terms due to ①.

• for $\mathbb{RP}^n$, also $w(\tau) = 1 + a$

because one has a bundle morphism

dimension of the base explicitly

$$\begin{tikzcd} \tau_1 \arrow[r] \arrow[d] & \tau_n \arrow[d] \\ \mathbb{RP}^1 \arrow[r, hook, "j"'] & \mathbb{RP}^n \end{tikzcd} \qquad (\tau_1 = j^* \tau_n)$$

- standard inclusion

$\Rightarrow$ $\underset{\substack{② \\ \neq 0}}{w_1(\tau_1)} = j^* w_1(\tau_n)$

$\Rightarrow w_1(\tau_n) \neq 0 \Rightarrow w_1(\tau_n) = a$

(and again by ① $w_{\geq 2} = 0$)

<u>Ex</u>: We have an inclusion

$$\begin{tikzcd} \tau \arrow[r, hook] \arrow[d] & \mathbb{R}^{n+1} \arrow[d] \\ \mathbb{RP}^n \arrow[r, "id"'] & \mathbb{RP}^n \end{tikzcd}$$

. Let $\underset{\substack{\downarrow \\ \mathbb{RP}^n}}{\overset{rk = n}{\tau^\perp}}$ - fiberwise orthog. complement of τ

Then $w(\tau^\perp) = w(\tau)^{-1} = (1+a)^{-1} = 1 + a + a^2 + \dots + a^n$

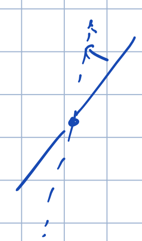
i.e. all w_j classes are $\neq 0$ for $j \leq n$

<u>Notation</u>: $\boxed{w(M) := w(TM)}$ - S-W class of the <u>tangent bundle</u> of a mfd

Thm 4.5 in M-S

<u>Thm</u> $w(\mathbb{RP}^n) = (1+a)^{n+1} = 1 + \binom{n+1}{1} a + \binom{n+1}{2} a^2 + \dots + \binom{n+1}{n} a^n$

<u>Idea of proof</u>: 1) $T\mathbb{RP}^n \cong \mathrm{Hom}(\tau, \tau^\perp)$



$v \in T_x \mathbb{RP}^n$
= infin. deformation of the line ℓ_x in orthog. direction
= graph of a lin map $\ell_x \to \ell_x^\perp$

2) $\mathrm{Hom}(\tau, \underbrace{\tau^\perp \oplus \tau}_{\mathbb{R}^{n+1}}) = \underbrace{\mathrm{Hom}(\tau, \tau^\perp)}_{T\mathbb{RP}^n} \oplus \underbrace{\mathrm{Hom}(\tau, \tau)}_{\substack{\text{triv. line bundle} \\ \text{(has a canonical} \\ \text{nonvanishing section)}}}$

$\quad \|$

$\underbrace{\mathrm{Hom}(\tau, \mathbb{R}) \oplus \dots \oplus \mathrm{Hom}(\tau, \mathbb{R})}_{n+1}$

$\quad \|$

$\tau^* \oplus \dots \oplus \tau^*$

$\quad \wr\|$ ← $\tau \cong \tau^*$ since τ has a Euclidean metric

$\tau \oplus \dots \oplus \tau$

$\Rightarrow w(\mathbb{RP}^n) = (1+a)^{n+1}$

$\square$

Ex:

n	$w(\mathbb{RP}^n)$	
0	1	
1	1	$\leftarrow (1+a)^2 = 1+2a+a^2$ evaluated in $\mathbb{Z}_2[a]/a^2=0$
2	$1+a+a^2$	$\leftarrow (1+a)^3 = 1+3a+3a^2+a^3$
3	1	$\leftarrow 1+4a+6a^2+4a^3+a^4$
4	$1+a+a^4$	$\leftarrow 1+5a+10a^2+10a^3+5a^4+a^5$
5	$1+a^2+a^4$	$\leftarrow 1+6a+15a^2+20a^3+15a^4+6a^5+a^6$
6	$1+a+a^2+a^3+a^4+a^5+a^6$	
7	1	$\Leftarrow: (1+a)^2 = 1+a^2,\ (1+a)^{2\cdot 2} = 1+a^4, \dots, (1+a)^{2^p} = 1+a^{2^p}$
...	...	

- $w(\mathbb{RP}^n) = 1$ iff $n = 2^p - 1$, $p = 0,1,2,3,\dots$

In fact, $T\mathbb{RP}^n$ is trivial iff $n \in \{0,1,3,7\}$ (cf. Theorem 4.7 in MS)

$\sim$ unit spheres in normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ are parallelizable

($\mathbb{H}$: quaternion, $\mathbb{O}$: octonion)

SW classes as obstructions to immersion

if $i: M \hookrightarrow \mathbb{R}^{n+k}$ immersion, then $w(NM) = w(M)^{-1}$ (Whitney duality thm)

(M: n-mfd)

$$\Rightarrow \quad \bar{w}_i(M) = 0 \quad \text{for } i > k$$

Ex:

n	$w(\mathbb{RP}^n)^{-1}$
0	1
1	1
2	$1+a$
3	1
4	$1+a+a^2+a^3$
5	$1+a^2$

e.g. for $\bar{w}_3(\mathbb{RP}^4) \neq 0$

$\Rightarrow$ $\mathbb{RP}^4$ cannot be immersed into $\mathbb{R}^{4+2}$

- Whitney Thm: any C^∞ cpt mfd M of dim $= n > 1$ can be immersed into $\mathbb{R}^{2n-1}$.
- This bound is saturated for $M = \mathbb{RP}^n$ for $n = 2^p$
- they cannot be immersed into $\mathbb{R}^{<2n-1}$