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On the Chern–Weil Homomorphism and the Continuous Cohomology of Lie-Groups

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1. INTRODUCTION

This note describes a formula found in collaboration with G. Hochschild during the spring of 1972 at Berkeley, California. Therefore I would like first to thank him for all he taught me during those months, and, second, to chide him for steadfastly refusing to share the blame with me as a joint author. Since that time independent work of Kamber Tondeur has also appeared [6], which covers much of this material but from a rather different point of view. (Thus Theorem 2.10 of their Manuscripta paper is essentially equivalent to our Theorem 1.)

The aim of our discussion was really to explain the Chern–Weil homomorphism to topologists, and I hope that in this respect we have succeeded. The present point of view also fits very well into the general framework of foliations and the continuous cohomology encountered there, as will be seen in a joint paper now in progress with A. Haefliger. However, in this note I will stick to the finite-dimensional case, and essentially start from scratch. To explain our formula recall first of all that the Chern–Weil homomorphism links the cohomology of the classifying space BG of a Lie-group, G, to the invariant forms on the Lie algebra $\mathfrak{g}$ of G. More precisely, if $\mathfrak{g}^*$ denotes the dual of $\mathfrak{g}$ as a G-module under the adjoint action, and $S\mathfrak{g}^*$ the symmetric algebra on $\mathfrak{g}^*$ in its induced G-module structure, then the Chern–Weil construction defines a homomorphism from the invariants $\mathrm{Inv}_G(S\mathfrak{g}^*)$ of $S\mathfrak{g}^*$ to $H^*(BG) = H^*(BG; \mathbb{R})$

$$\varphi\colon \mathrm{Inv}_G(S\mathfrak{g}^*) \to H^*(BG). \tag{1.1}$$

The construction of this φ was inspired by differential geometric considerations and generalizes earlier constructions for the Pontryagin

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and Chern classes from the curvature of a connection. Partly because of these geometric antecedents this homomorphism φ has also been treated with some apprehension by topologists in general. In a recent thesis [7], Shulman constructs φ more in the spirit of the algebraic topology and this note answers a question which arises naturally in his framework.

To motivate our main result let me review here the very elementary point of view of "characteristic classes in terms of transition functions" which I have advocated for some time now (see [1], [2]) and which is also the starting point of Schulman's thesis.

In this framework we think of a principal G-bundle ξ as being described by its 1-cocycle $g = \{g_{UV}\}$ of smooth transition functions

$$g_{UV}: U \cap V \to G$$

defined on the overlaps of open sets of a covering $\mathfrak{U}$ of the base manifold M of ξ, and we seek characteristic classes of ξ in the Cěch–DeRham complex," $\check{C}(\mathfrak{U}; A)$, of M *relative to* $\mathfrak{U}$. Thus $\check{C}(\mathfrak{U}; A) = \Sigma_{p,q} C^p(\mathfrak{U}; A^q)$ where $C^p(\mathfrak{U}; A^q)$ are the Cěch p-cochains relative to the cover $\mathfrak{U}$, in the presheaf of q-forms on M. This complex has two differential operators, the d of the DeRham theory, and the δ of the Cěch theory, which combine with a suitable sign, to yield a total differential $D = d \pm \delta$. The homology $H\check{C}(\mathfrak{U}; A)$, of $\check{C}(\mathfrak{U}; A)$ relative to this D figures already in A. Weil's 1951 proof of DeRham's theorem, the important point being that for every open cover $\mathfrak{U}$ the natural inclusion of $A(M) \subset \check{C}^0(\mathfrak{U}; A)$ induces an isomorphism of the DeRham groups of M with $H\check{C}(\mathfrak{U}; A)$:

$$HAM \simeq H\check{C}(\mathfrak{U}; A). \tag{1.2}$$

This granted it suggests itself to construct characteristic classes for ξ, via a cocycle g for ξ over $\mathfrak{U}$, in $\check{C}(\mathfrak{U}; A)$ by the following procedure: Once and for all we choose differential forms $\varphi = \{\varphi_p\}$, $p = 0, 1$, with

$$\varphi_p \in A(G \times \cdots \times G) \qquad p \text{ factors.}$$

Next, given any cocycle g, of ξ valid on $\mathfrak{U}$, and any nontrivial intersection

$$U_\sigma = U_0 \cap \cdots \cap U_p$$

of $(p + 1)$ open sets in $\mathfrak{U}$, let

$$g_\sigma: U_\sigma \to G \times \cdots \times G \qquad (p \text{ factors})$$

be the product map

$$g_\sigma = g_{U_\sigma U_1} \times g_{U_1 U_2} \times \cdots \times g_{U_{p-1} U_p}.$$

Then for each σ, $g_\sigma{}^*\varphi_p$ is a well defined form on U_σ, and the collection of these forms clearly defines a cochain φ

$$g^*\varphi \in \check{C}(\mathfrak{U}, A).$$

One is next led to ask when $g^*\varphi$ is *universally closed* in $\check{C}(\mathfrak{U}, A)$ (relative to D).

For this purpose consider the naturally bigraded complex $\mathcal{O}\!\mathcal{l}(G) = \Sigma\mathcal{O}\!\mathcal{l}^{p,q}(G)$ with

$$\mathcal{O}\!\mathcal{l}^{p,q}(G) = A^q(G \times \cdots \times G) \qquad p \text{ factors}, \tag{1.3}$$

so that we may then think of φ as an element of $\mathcal{O}\!\mathcal{l}G$.

Now of course $\mathcal{O}\!\mathcal{l}G$ has the differential d of DeRham theory, but can also be equipped with a differential operator δ

$$\delta\colon \mathcal{O}\!\mathcal{l}^{q,p}(G) \to \mathcal{O}\!\mathcal{l}^{q,p+1}(G)$$

which commutes with d and is compatible with the Cěch δ of $\check{C}(\mathfrak{U}; A)$ under g^*. In short then, the complex $\mathcal{O}\!\mathcal{l}(G)$ also has a natural *total operator D* defined on it and the cochain *$g^*\varphi$ will be universally* closed, *if and only if φ is D*-closed in $\mathcal{O}\!\mathcal{l}(G)$.

At this stage one is therefore naturally led to consider the homology of $\mathcal{O}\!\mathcal{l}(G)$ relative to D as furnishing one with the simplest characteristic classes of G-bundles, and to compute $H\mathcal{O}\!\mathcal{l}(G)$ explicitly, there are the standard methods of treating double complexes. Namely, one filters $\mathcal{O}\!\mathcal{l}(G)$ either by p, or by q, and considers the resulting spectral sequences.

If one filters by $\leqslant p$, the resulting spectral sequence obviously has the E_1-term

$$E_1^{p,q} = H^q(\underbrace{G \times \cdots \times G}_{p})$$

and the d_1, induced by δ, in $A(G)$, is then seen to yield the cobar construction on the coalgebra $H^*(G)$. Furthermore, it then follows without trouble that the E_2 term of this sequence is the symmetric algebra generated by the *primitive* elements $\mathscr{P}(G)$ of $H(G)$, considered in $H^{1,*}(G)$ and that all the further differential operators vanish. All this is done by Schulman in [7], and clearly leads to the evaluation of $H\mathcal{O}\!\mathcal{l}G$:

$$H\mathcal{O}\!\mathcal{l}(G) \cong S\mathscr{P}(G). \tag{1.4}$$

This is of course the "correct result" in the sense that the characteristic classes predicted by the classification theorems are given by the cohomology of the classifying space BG of G, and this cohomology is, over the reals, well known to be a polynomial algebra on the primitive elements of $H^*(G)$. Furthermore, this is *not* coincidence; indeed the spectral sequence obtained above is very naturally related to the sequences obtained from any of the standard procedures for constructing BG, see [7] for details.

Note finally that up to this point the precise nature of the q-form functor A^q, is nowhere brought out essentially. For instance if we interpret $A^q(U)$ as singular cochains on U, (1.4) would still be valid by the same argument.

Now our main observation is that the special nature of the DeRham functor A^q, can be exploited to yield an explicit evaluation of the cohomology of $\mathcal{O}\!\mathcal{l}(G)$ relative to δ which in turn leads to a generalization of the Chern–Weil homomorphism. Precisely we have:

THEOREM 1. *Let G be a Lie-group and consider the δ-closed subcomplex, $\mathcal{O}\!\mathcal{l}^qG$, of $\mathcal{O}\!\mathcal{l}G$ determined by the q forms. Thus*

$$\mathcal{O}\!\mathcal{l}^qG\colon A^q(p) \xrightarrow{\delta} A^q(G) \xrightarrow{\delta} A^q(G \times G) \cdots .$$

Then the δ-homology of $\mathcal{O}\!\mathcal{l}^qG$ is given by

$$H_\delta{}^p\mathcal{O}\!\mathcal{l}^qG = H_c^{p-q}(G; S^q\mathfrak{g}^*), \tag{1.5}$$

where $\mathfrak{g}$ is the Lie algebra of G considered as a G-module under the adjoint action, $S^q\mathfrak{g}^*$ denotes the q-th symmetric power, and the subscript c denotes the "smooth" or equivalently "continuous" cohomology of G with values in $S^q\mathfrak{g}^*$.

Remarks. (a) That (1.5) is connected to the Chern–Weil homomorphism is clear once one recalls the first facts concerning the continuous cohomology. These are that:

$$H_c{}^0(G; W) = \mathrm{Inv}_G\, W, \tag{1.6}$$

and that

$$H_c{}^i(G, W) = 0, \qquad \text{for} \quad i > 0, \tag{1.7}$$

if G is compact.

Thus by (1.5)

$$H_\delta{}^p\mathcal{O}\!\mathcal{l}^pG = \mathrm{Inv}_G\, S^p\mathfrak{g}^*,$$

whence, as there is no cohomology above the diagonal, the "edge-homomorphism" of (1.5) induces a natural arrow,

$$\Phi\colon \mathrm{Inv}_G(S\mathfrak{g}^*) \to H\mathcal{O}\!\mathcal{C}G$$

which, under the identification $H^*\mathcal{O}\!\mathcal{C}G \simeq H^*BG$ goes over into the Chern–Weil homomorphism. Further by (1.7) Φ is *an isomorphism* when G is compact. The formula (1.5) therefore easily yields the classical properties of φ, and at the same time clarifies the situation in the non-compact case. Further, our procedure involves no "curvature" arguments. Rather, one encounters the homomorphism Φ as a direct consequence of the DeRham functor A^q.

(b) On the debit side it should be pointed out that Theorem 1—although we could not find it explicitly in the literature— is not really new, but in some sense a specialization of theorems of Illusie outlined in [2], on Chern classes in the algebraic category. However, as this paper is highly inaccessible to most geometers and topologists, and as continuous cohomology does not really fit into the algebraic theory a direct derivation of this theorem, such as we bring below, might be of use in any case.

There are also some very recent notes by Kamber and Tondeur [4], which are closely related to (1.5) via the Van Est Theorem.

The proof of our theorem, which we bring in the next section, fits very naturally into the Dold-Puppe Yoga of nonadditive functors [3] and especially their notion of semisimplicial derivation of a functor F defined on the category of modules. Indeed, the shift from the *exterior algebra* of the DeRham theory to the *symmetric algebra* of the Chern–Weil homomorphism will be seen to correspond precisely to the following fact.

PROPOSITION 1.8. *In characteristic* 0 *the first derived* (*in the sense of Dold-Puppe*) *of the exterior power* Λ^q, *is given by the q-th syspension of* S^q, *the q-th symmetric power.*

For completeness sake we will review some of the notions from scratch in the next section, before proving this proposition. The proof of Theorem 1 is then brought in Section 3, with the aid of the Hochschild–Mostow characterization of the continuous cohomology of G.

2. $\mathcal{O}\!\mathcal{C}G$ AS A SEMI-SIMPLICIAL OBJECT

The simplest description of the complex $\mathcal{O}\!\mathcal{C}G$ described in the introduction is in semi-simplicial terms. Indeed, in that framework,

$\mathfrak{A}G$ is seen to be derived from a *semi-simplicial homogeneous space* NG, associated to G, by first applying the DeRham functor to obtain a *cosemisimplicial module* ANG, and thereafter the functor k converting such objects into cochain complexes. Thus one has the factorization

$$\mathfrak{A}G = k \circ A \circ NG. \tag{2.1}$$

Precisely, let Ord denote the category of finite linearly ordered sets, with order preserving maps, and denote as is usual, the ordered set $\{0 < 1 < \cdots < n\}$ by $\mathbf{n}$. A semisimplicial set, space, manifold, module, group, etc., is then by definition a contravariant functor from Ord to the category of sets, spaces, manifolds, groups, modules, etc. Covariant functors of this type are unfortunately called cosemisimplicial sets, spaces, etc.

Note that if X is an object in one of these categories, then X defines *two* "trivial" semisimplicial objects of that same category which we denote by X again, and PX respectively.

The semisimplicial object X is simply defined by:

$$X(\mathbf{n}) = X, \qquad \text{all maps, } \to\text{, the Identity,} \tag{2.2}$$

while PX is defined in terms of the underlying sets $\mathbf{n}^\natural$ and $X^\natural$, of $\mathbf{n} \in \text{Ord}$ and X by:

$$PX(\mathbf{n}) = \text{Hom}(\mathbf{n}^\natural, X^\natural). \tag{2.3}$$

Less pedantically, this of course means that

$$PX(\mathbf{n}) = X \times \cdots \times X \qquad (n + 1 \text{ copies}).$$

Note also that if X is a manifold, group, etc. PX is a semisimplicial manifold, group, etc., in a natural manner.

Note further that the diagonal maps $\Delta\colon X \to X \times \cdots \times X$ combine to define a natural morphism

$$X \xrightarrow{\Delta} PX. \tag{2.4}$$

Now suppose G is a group. Then $PG(\mathbf{n})$ is a group for each $\mathbf{n}$, and $\Delta\colon G(\mathbf{n}) \to PG(\mathbf{n})$ is the diagonal inclusion. We may therefore define a new functor NG by setting

$$NG(\mathbf{n}) = PG(\mathbf{n})/G(\mathbf{n}), \tag{2.5}$$

and if G is a Lie-group, NG is clearly a semisimplicial manifold. The DeRham functors A^q, are therefore well defined on $NG(\mathbf{n})$, and the composition $A^q \circ NG$, defines the cosemisimplicial object:

$$A^q \circ NG\colon \mathrm{Ord} \to \mathbb{R}\text{-modules}. \tag{2.6}$$

At this stage one needs only explain the functor k to make sense of the right-hand side of (2.1). For this purpose let M: Ord $\to \mathscr{M}$ be any covariant functor with values in modules. Then kM denotes the cochain complex with the module $M(\mathbf{n})$ in dimension n, and differential operator

$$\delta\colon M(\mathbf{n}) \to M(\mathbf{n}+1)$$

defined by:

$$\delta = \sum (-1)^i M(\epsilon_i) \tag{2.7}$$

where $\epsilon_i\colon \mathbf{n} \to \mathbf{n}+1$, $i = 0,\ldots\ n+1$ is the unique monotone map whose image does not include i. With this understood we have the following straightforward proposition which we leave to the reader.

PROPOSITION 2.8. *The formula* (2.1) *is valid. That is the cochain complex $k \circ A^q \circ NG$ can be identified with the complex $\mathcal{O}\!\mathcal{l}^q G$.*

To proceed further we will need to decompose $A^q \circ NG$ a good deal more. In particular we will want to bring out clearly the role of the exterior power in this functor and to do this we will need the following two auxiliary constructions.

Given two cosemisimplicial modules M and N, their product $M \times N$ is defined to be the functor of the same sort given by

$$(M \times N)(\mathbf{n}) = M(\mathbf{n}) \otimes N(\mathbf{n}). \tag{2.9}$$

Also given a module M one has constructions CM and ΣM, analogous to the earlier PG and NG constructions. Explicitly let us first agree to identify a module M with the trivial functor

$$M\colon \mathrm{Ord} \to \mathscr{M}$$

which assigns M to all the objects $\mathbf{n}$ of Ord and sends all maps into 1. Next we take the forgetful functor $\mathbf{n} \to \mathbf{n}^\natural$ and convert it to the covariant functor

$$C\mathbb{Z}\colon \mathrm{Ord} \to \mathscr{M}$$

by setting

$$C\mathbb{Z}(\mathbf{n}) = \mathbb{Z}(n^\natural). \tag{2.10}$$

Here $\mathbb{Z}$(Set) denotes the free group generated by the set. One now has a natural arrow

$$C\mathbb{Z} \xrightarrow{r} \mathbb{Z} \longrightarrow 0$$

defined by

$$r(\mathbf{n})\Big(\sum_{\alpha=0,\ldots n} a_\alpha \otimes \alpha\Big) = \sum a_\alpha\,, \qquad a_\alpha \in \mathbb{Z},$$

The kernel of r therefore defines a new functor $\Sigma\mathbb{Z}$ which we call the "suspension of $\mathbb{Z}$" and which naturally fits into the exact sequence:

$$0 \to \Sigma\mathbb{Z} \to C\mathbb{Z} \to \mathbb{Z} \to 0.$$

More generally if M is any covariant functor from Ord to modules, we define its cone and suspension CM and ΣM, by

$$\begin{aligned} CM &= C\mathbb{Z} \otimes M \\ \Sigma M &= \Sigma\mathbb{Z} \otimes M, \end{aligned} \qquad \text{(tensor over } \mathbb{Z}) \tag{2.11}$$

and one has the natural exact sequence:

$$0 \to \Sigma M \to CM \to M \to 0. \tag{2.12}$$

The reason for this terminology is of course that $H(kC\mathbb{Z})$ and hence $H(kCM)$ vanishes, whence if we abbreviate $H(kM)$ to $H(M)$:

$$H(\Sigma M) = M \qquad \text{(in dimension 1)},$$

that is, $H^i(\Sigma M) = 0$ for $i \neq 1$ and $H^1(\Sigma M) = M$ for any module M.

With this notation understood, we now have the following *basic decomposition* of A^qNG:

DECOMPOSITION LEMMA. *Let G be a Lie-group, and let $\mathfrak{g}^*$ be the $\mathbb{R}$-module of left-invariant 1-forms on G. We make $\mathfrak{g}^*$ into a G-module under the action induced by right multiplication, make $A^0 \circ PG$ into a G-module valued functor through the action of G on $PG(\mathbf{n})$ given by right diagonal multiplication, and let G act trivially on $\Sigma\mathbb{Z}$.*

So understood, the functor $A^q \circ NG$ has the decomposition:

$$A^q \circ NG \cong \mathrm{Inv}_G\{A^0 \circ PG \times \Lambda^q(\Sigma\mathbb{Z} \times \mathfrak{g}^*)\}. \tag{2.13}$$

This formidable expression is actually just a very pedantic description of how the forms on the base of a fibering

$$E \xrightarrow{\pi} B,$$

appear in the complex of forms on the total space when both the tangent space to E and the normal space to the fibers of π admit trivializations,

Indeed let $\nu(\pi)$ be the subbundle of the cotangent bundle $T_E{}^*$ to E, whose elements restrict to 0 on the fibers of π. Then under π^*, $A^q(B)$ is clearly mapped isomorphically onto the space $\Gamma_F(\Lambda^q\nu^*)$ of those C^∞ sections of $\Lambda^q\nu^*$ which are constant on each fiber relative to the trivialization of ν^*/F induced by π^*

$$\pi^*\colon A^q(B) \cong \Gamma_F(\Lambda^q\nu^*).$$

Now suppose in addition that ν^* admits a global trivialization

$$\alpha\colon \nu^* \to W \tag{2.15}$$

with a fixed vector space W. Such an α identifies $\Gamma(\nu^*)$ with the W-valued functions on E, that is, with $A^0(E) \otimes W$ and $\Gamma_F(\nu^*)$ then becomes identified with a certain submodule of $A^0(E) \otimes W$. In our situation we apply this principle to the projection

$$PG(\mathbf{n}) \xrightarrow{\pi} NG(\mathbf{n})$$

and the trivialization of $\nu^*(\pi)$ induced by the left invariant forms on G. Indeed the inclusion

$$\iota\colon \mathfrak{g}^* \to A^1G,$$

extends to an arrow

$$\iota(\mathbf{n})\colon C\mathfrak{g}^*(\mathbf{n}) \to A^1PG(\mathbf{n}) \tag{2.16}$$

by the formula

$$\iota(\mathbf{n}) \sum_{k=0}^{n} a_k \otimes x_k = \sum a_k \cdot \pi_k{}^* x_k$$

where the π_k: $PG(\mathbf{n}) \to G$ are the natural projections. Now because of the left invariance of the forms the image of any element in $C\mathfrak{g}^*$ under $\iota(\mathbf{n})$ restricts to the *same* left invariant form on *every fiber* of π. That is, under the map

$$G \to PG(\mathbf{n})$$

given by $g \to (p_0 \cdot g \times p_1 \cdot g \times \cdots \times p_n \cdot g)$, $p_i \in G$, the element $\sum a_\alpha \pi_\alpha{}^* x_\alpha$ maps to $\sum a_\alpha x_\alpha \in \mathfrak{g}^*$. Hence under $\iota(\mathbf{n})$ the kernel $\sum \mathfrak{g}^*(\mathbf{n})$ (as defined in (2.12)) maps into $\nu^*(\pi)$, and in fact *induces a trivialization* of $\nu^*(\pi)$, because the image of $C\mathfrak{g}^*(\mathbf{n})$ under $\iota(\mathbf{n})$ spans the cotangent space of $PG(\mathbf{n})$ at every point.

In the resulting identification of $\Gamma(\nu^*)$ with $A^0(G) \otimes \sum \mathfrak{g}^*(\mathbf{n})$ it is now clear that the image of A^1NG is precisely given by the G-invariant elements, and more generally

$$A^q NG(\mathbf{n}) \simeq \mathrm{Inv}_G \{A^0 PG(\mathbf{n}) \otimes \Lambda^q \Sigma \mathfrak{g}^*(\mathbf{n})\}$$

as was to be shown.

3. The Proof of Theorem 1

The decomposition formula of the last section clearly yields the result

$$\mathfrak{A}^q G = \mathrm{Inv}_G [k \{A^0 PG \times \Lambda^q \Sigma \mathfrak{g}^*\}]. \tag{3.1}$$

Now consider the cochain complex

$$\mathscr{C}^q G = k(A^0 PG \times \Lambda^q \Sigma \mathfrak{g}^*), \tag{3.2}$$

and recall the Cartier extension of the Eilenberg–Zilber Theorem [3] which asserts that for any two cosemisimplicial modules M and N, the cochain complexes $k(M \times N)$ and $k(M) \otimes k(N)$ are naturally homotopic:

$$k(M \times N) \sim k(M) \otimes k(N). \tag{3.3}$$

Hence, in our characteristic 0 case,

$$H(\mathscr{C}^q G) \simeq H(A^0 PG) \otimes H(\Lambda^q \Sigma \mathfrak{g}^*). \tag{3.4}$$

We now assert that

LEMMA 3.1. *$H(A^0PG) = \mathbb{R}$ (in* dim 0) *while*

$$H(\Lambda^q\, \Sigma\mathfrak{g}^*) = S^q\mathfrak{g}^* \qquad (\textit{in } \dim q). \tag{3.5}$$

The first assertion is well known and corresponds to the contractability of the total space in the bar construction. The second one which is equivalent to Proposition 1.8 of the Introduction, is also rather clear from topological considerations and most probably known to experts. In any case we will bring proofs in the next section. But assuming this lemma, we can proceed to the proof of Theorem 1 as follows.

Clearly (3.4) and the lemma imply that

$$H\mathscr{C}^qG = S^q\mathfrak{g}^* \qquad (\text{in } \dim q).$$

On the other hand one checks directly that qua-G-modules the cochain complex $\mathscr{C}$ is *continuously injective* in the sense of [5].

Equation (3.5) therefore asserts that up to a shift of q to the right; $\mathscr{C}^qG$ is a *continuously injective resolution* of $S^q\mathfrak{g}^*$. Hence by the very definition of continuous cohomology, à la Hochschild–Mostow,

$$H(\mathscr{A}^qG) = H_c(G; S^q\mathfrak{g}^*)$$

with a shift in dimension of q; that is

$$H^p(\mathscr{A}^qG) = H_c^{p-q}(G; S^q\mathfrak{g}^*),$$

as was to be shown.

4. SEMISIMPLICIAL SUSPENSION AND THE PROOF OF LEMMA 3.1

The first assertion of our lemma, namely that

$$H(A^0 \circ PG) \simeq \mathbb{R} \qquad (\text{in } \dim 0),$$

of course amounts to the classic cone construction, and holds for any object X. That is, given an X in one of the categories discussed, then

$$H(A^0 \circ PX) = \mathbb{R} \qquad (\text{in } \dim 0) \tag{4.1}$$

for "any" interpretation of A^0 as arbitrary, continuous, or smooth functions on the objects. Indeed one merely chooses a point $e \in X$, and in terms of it defines a contracting homotopy on kA^0PX by the formula:

$$\lambda\varphi(x_0, \ldots, x_{p-1}) = \varphi(e, x_0, \ldots, x_{p-1}).$$

The formula (3.6) is less trivial, but as we will show follows directly from Eilenberg–Zilber–Cartier and the analog of (4.1) for a semi-simplicial module M, that is, the formula:

$$H(CM) = 0. \tag{4.2}$$

To see (4.2) one may use the following construction due to Dold and Puppe. Given any cosemisimplicial module M, they associate to M a cochain complex $k^\flat M$ which is much smaller than kM *but has the same cohomology*, by setting

$$k^\flat M_n = M(\mathbf{n}) \Big/ \Big\{ \sum_{0<i} M(\epsilon_i)\, M(\mathbf{n}-1) \Big\}$$

and letting the differential operator of $k^\flat M$ be the one induced by $\delta_0 = M(\epsilon_0)$ on $k^\flat M$ under the natural projection:

$$kM \xrightarrow{\flat} k^\flat M.$$

A powerful computational device of [3] is the proposition that $\flat$ induces an isomorphism:

$$H(kM) \simeq H(k^\flat M). \tag{4.3}$$

Now consider the object CM for M a fixed module: This object "starts" with

$$M \rightrightarrows M \oplus M \substack{\rightarrow\\ \rightarrow\\ \rightarrow} M \oplus M \oplus M \quad \text{etc.}$$

whence $k^\flat CM$ is easily seen to reduce to

$$M \xrightarrow{\delta_0} M \longrightarrow 0 \longrightarrow 0$$

with δ_0 the identity. Q.E.D.

Thus kCM is an "augmented cone" with no homology in any dimension, whence it follows directly from the exact sequence

$$0 \to \Sigma M \to CM \to M \to 0,$$

that

$$H(k\,\Sigma M) = M \qquad \text{(in dimension 1)}. \tag{4.4}$$

Now suppose F is any functor from modules to modules. In [3] Dold and Puppe define the semisimplicial suspension or derivation LF, of F, which is a functor from semisimplicial modules to graded modules, obtained by first syspending and then taking homology. On a fixed module M (or equivalently on the trivial cosemisimplicial object defined by M) LF can therefore be defined explicitly by

$$LF(M) = H(F \circ \Sigma M), \tag{4.5}$$

and the assertion (3.5) of Lemma 3.1 is equivalent to the assertion that

$$L\Lambda^q(M) = S^q(M) \qquad \text{(in dim } q). \tag{4.6}$$

We will prove (4.6) in a slightly more general setting which explains the operation L on all functors F, obtained by "symmetrization" from the functor

$$T_q \colon M \to M \otimes \cdots \otimes M \qquad (q \text{ copies}). \tag{4.7}$$

First of all observe that the theorem of Eilenberg–Zilber–Cartier immediately evaluates LT_q. Indeed

$$T_q(M) = M \times \cdots \times M \qquad (q \text{ copies})$$

for any cosemisimplicial object. Hence

$$T_q\,\Sigma M = \Sigma M \times \cdots \times \Sigma M \qquad \text{for any module } M.$$

It therefore follows from Eilenberg–Zilber–Cartier plus characteristic 0 that

$$\begin{aligned} LT_q(M) = H(kT_q\,\Sigma M) &= H(k\,\Sigma M) \otimes \cdots \otimes H(k\,\Sigma M) \\ &= M \otimes \cdots \otimes M \qquad \text{(in dim } q). \end{aligned} \tag{4.8}$$

Thus for T_q, the operation L simply "moves T_q up by q steps":

$$LT_q \cong T_q \qquad \text{(in dim } q). \tag{4.9}$$

Now this formula admits a refinement which in addition clears up how, at least in characteristic 0, the subfunctors of T_q, such as Λ^q, S_q, etc. behave under L. To formulate it we first "promote" T_q to a functor from $\mathbb{R}$-modules to modules over the group ring $\mathbb{R}\mathbf{S}_q$ of the symmetric group on q letters.

The action of $\mathbb{R}\mathbf{S}_q$ on T_q is of course given by the permutation of the factors. We also denote by $\bar{T}_q$ the functor $M \to M \otimes \cdots \otimes M$ (q factors) on which $\mathbb{R}\mathbf{S}_q$ acts by twisting the operation of $\mathbf{S}_q$ on T_qM with the determinant character:

$$\sigma \to \det \sigma$$

Thus in $\bar{T}_q(M)$,

$$\sigma(m_1 \otimes \cdots \otimes m_q) = \det \sigma(m_{\sigma^{-1}(1)} \otimes m_{\sigma^{-1}(2)} \otimes \cdots \otimes m_{\sigma^{-1}(q)}).$$

An immediate consequence of $E - Z - C$ is now the

PROPOSITION 4.10. *Considered as functors from $\mathbb{R}$-modules to $\mathbb{R}\mathbf{S}_q$-modules, the q-th powers T_q and $\bar{T}_q$ are "interchanged" by L. That is,*

$$LT_q(M) = \bar{T}_qM \qquad (\text{in dim } q)$$

$$L\bar{T}_q(M) = T_qM \qquad (\text{in dim } q).$$

Indeed this is just a consequence of the well known fact that under an equivalence

$$k(X \times X) \sim k(X) \otimes k(X)$$

the interchange of factors on the left is expressed, on the right, by the chain map $u^p \otimes v^q \to (-1)^{pq} v^q \otimes u^p$; u and v in dim p and q, respectively. Now in our situation $k(\Sigma M)$ has homology only in dim 1, whence all transpositions pick up the sign -1. Q.E.D.

Consider now the *subfunctors* of T_q given by symmetrization with respect to a central idempotent e in $\mathbb{R}\mathbf{S}_q$. That is, given e with $e^2 = 1$, $e\sigma = \sigma e$ all $\sigma \in \mathbf{S}_q$ define

$$F_e(M) = e \cdot T_qM.$$

We clearly have arrows $F_e(M) \underset{i}{\overset{e}{\leftrightarrows}} T_q(M)$ given by the inclusion and the projection e respectively, which induce a direct sum decomposition

$$HkF_e \sum M \leftrightarrows HkT_q \sum M = \bar{T}_q(M) \qquad (\text{in dim } q). \tag{4.11}$$

Hence the left-hand side of (4.11) is given by $e\bar{T}_qM$ (in dim q). Thus one has the formula:

$$LF_e(M) \cong e\bar{T}_qM \qquad (\text{in dim } q). \tag{4.12}$$

On the purely module level this can also be expressed as follows:

The group ring $\mathbb{R}\mathbf{S}_q$ admits an automorphism denoted by —, given by

$$\bar{\sigma} = \det \sigma \cdot \sigma,$$

and $\bar{T}_qM$ is clearly obtained from T_qM via this automorphism. Hence qua vector spaces, we have

$$e \cdot \bar{T}_qM = \bar{e}T_qM, \tag{4.13}$$

so that on this level we can write (4.12) in the form

$$LF_e(M) \cong F_{\bar{e}}(M) \qquad (\text{in dim } q). \tag{4.14}$$

Finally, as the bar clearly interchanges alternation with symmetrization, this yields the desired formula

$$L\Lambda^q \simeq S^q \qquad (\text{in dim } q).$$

Also note that in the language of Young diagrams (it can be shown that) the bar corresponds to exchanging the rows with the columns in a diagram.

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