INTERMEDIATE GEOMETRY AND TOPOLOGY EXERCISES 1, 8/27/2021

1. Consider a real line bundle E over $S^1 = \mathbb{R}/\mathbb{Z}$, defined as follows: cover S^1 by 3 open intervals $U_j = \left(\frac{j-1}{3} - \epsilon, \frac{j}{3} + \epsilon\right)$, with $j = 1, 2, 3$, for some positive $\epsilon < \frac{1}{6}$. Define the transition functions to be constant functions on the overlaps $t_{12} = +1,\ t_{23} = +1,\ t_{31} = -1$. Prove that the bundle E is not isomorphic to the trivial bundle. Prove that if instead we choose $t_{12} = +1,\ t_{23} = -1,\ t_{31} = -1$, then the bundle is isomorphic to the trivial one.

2. It can happen that two vector bundles can be connected by an isomorphism, but not by one covering the identity on the base. Here is an example. Denote MB the Möbius band regarded as a line bundle over S^1. Also, let $\underline{\mathbb{R}}$ be the trivial line bundle over S^1. Denote $\pi_{\mathrm{MB}}, \pi_{\mathrm{triv}}$ the corresponding bundle projections to S^1. Consider two vector bundles

$$\begin{array}{c} E_1 = \mathrm{MB} \times \underline{\mathbb{R}} \\ \Big\downarrow{\scriptstyle \pi_{\mathrm{MB}} \times \pi_{\mathrm{triv}}} \\ S^1 \times S^1 \end{array} \quad , \qquad \begin{array}{c} E_2 = \underline{\mathbb{R}} \times \mathrm{MB} \\ \Big\downarrow{\scriptstyle \pi_{\mathrm{triv}} \times \pi_{\mathrm{MB}}} \\ S^1 \times S^1 \end{array}$$

and $E_2 = \underline{\mathbb{R}} \times \mathrm{MB}$ over $S^1 \times S^1$ Show that the one can find an invertible morphism of bundles from E_1 to E_2, but not one covering the identity map on the base $S^1 \times S^1$.

3. Let E be a (real, smooth) vector bundle over a manifold M of rank k. Prove that E is isomorphic to a trivial bundle if and only if one can find a k-tuple of sections $\sigma_1, \ldots, \sigma_k$ of E such that at each point $x \in M$, vectors $\sigma_1(x), \ldots, \sigma_k(x)$ in the fiber E_x are linearly independent.

4. (a) Show that for M a compact manifold with nonzero Euler characteristic, the tangent bundle TM is not trivial.[1]
   (b) Show that the tangent bundle TS^2 of the 2-sphere is nontrivial but its Whitney sum with the trivial rank one bundle $TS^2 \oplus \underline{\mathbb{R}}$ is trivial.[2]
   (Generally, a bundle E such that $E \oplus \underline{\mathbb{R}}^k$ is trivial for some $k \geq 0$ is called "stably trivial." Thus, TS^2 is nontrivial but stably trivial, with $k = 1$.)

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[1] Hint – use Poincaré-Hopf theorem: for any vector field v on M with isolated zeroes, $\chi(M) = \sum_i \mathrm{index}_{x_i}(v)$ with the sum going over zeroes x_i of v.

[2] Hint: note that $\underline{\mathbb{R}}$ is isomorphic to the normal bundle of the embedding $S^2 \hookrightarrow \mathbb{R}^3$.

5. (a) Calculate the transition functions for the tautological line bundle τ over $\mathbb{CP}^1$. Use the standard "stereographic" cover of $\mathbb{CP}^1$ by open sets $D_+ = \{(1 : z) | z \in \mathbb{C}\}$ and $D_- = \{(w : 1) | w \in \mathbb{C}\}$.
   (b) What are the transition functions for $\tau^{\otimes n}$? For τ^*?
   (c) Consider the "canonical bundle" of $\mathbb{CP}^1$ – the complex line bundle $K = (T^{1,0})^*\mathbb{CP}^1$ whose sections are the differential forms of the form $f(z)dz$ in D_+ and $g(w)dw$ in D_-. Show that K is isomorphic to $\tau^{\otimes 2}$.

6. Show that the pullback of the tautological bundle over $\mathbb{RP}^2$ along the covering map $p \colon S^2 \to \mathbb{RP}^2$ is trivial.