INTERMEDIATE GEOMETRY AND TOPOLOGY EXERCISES 2, 9/3/2021. UPDATED VERSION.

1. Consider the Hopf fibration $S^3 \to \mathbb{CP}^1$ as a principal S^1-bundle. Show that the associated complex line bundle (using the standard action of $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ on $\mathbb{C}$ by multiplication) is isomorphic to the tautological line bundle τ over $\mathbb{CP}^1$.

2. Consider an Ehresmann connection in a trivial vector bundle $\pi\colon U \times \mathbb{R}^k \to U$ given by a horizontal distribution assigning to a point $(x, v) \in U \times \mathbb{R}^k$ the subspace $H_{(x,v)} = \mathrm{graph}(-A \circ v) \subset T_{(x,v)}(U \times \mathbb{R}^k)$. Here $A \in \Omega^1(U, \mathfrak{gl}(k))$ – a matrix-valued 1-form on the base.

   (a) Show that the corresponding covariant derivative operator[1] $\nabla\colon \Omega^p(U, \underline{\mathbb{R}}^k) \to \Omega^{p+1}(U, \underline{\mathbb{R}}^k)$ acts as $\nabla = d + A$.[2]

   (b) Show that the curvature 2-form on the total space $\mathcal{F} \in \Omega^2(U \times \mathbb{R}^k, V)$, defined by[3]

$$\mathcal{F}(X, Y) = -[X_H, Y_H]_V \tag{1}$$

   for X, Y vector fields on the total space, has the form $\mathcal{F} = \pi^* F$ where

$$F = dA + \frac{1}{2}[A, A] \quad \in \Omega^2(U, \mathfrak{gl}(k)) \tag{2}$$

   – a matrix-valued 2-form on the base.[4]

3. Consider the trivial principal G-bundle $\mathcal{P} = U \times G$ over U with G a matrix Lie group. Assume the bundle $\mathcal{P}$ is equipped with a connection defined by a 1-form $\mathcal{A} \in \Omega^1(\mathcal{P}, \mathfrak{g})$ on the total space. Let $\sigma\colon x \mapsto (x, 1)$ be the unit section of $\mathcal{P}$ and

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[1] Recall that we define ∇ on sections by $(\nabla\sigma)(v) = \frac{d}{dt}\Big|_{t=0} (\mathrm{Hol}_{\gamma_0^t})^{-1}\sigma(\gamma(t))$ where $v \in T_xU$ is a tangent vector on a base at a point $x \in U$, γ is any curve $\gamma\colon [0, 1] \to U$ satisfying $\gamma(0) = x$, $\dot\gamma(0) = v$. $\mathrm{Hol}_{\gamma_0^t}\colon \underbrace{E_x}_{\mathbb{R}^k} \to \underbrace{E_{\gamma(t)}}_{\mathbb{R}^k}$ is the parallel transport along the stretch of the curve γ from time 0 to time t.

[2] First consider the case $p = 0$, i.e., show that ∇ maps a section σ (understood as a vector-valued function on U) to $d\sigma + A \circ \sigma$, then extend the result to $p > 0$ by Leibniz identity.

[3] I think there should be a minus sign in the definition of Ehresmann curvature (1) for consistency with other sign conventions.

[4] Hint: one approach is to use a local coordinate chart on the base and write vector fields on the total space as $X = X^\mu(x, v)\frac{\partial}{\partial x^\mu} + X^i(x, v)\frac{\partial}{\partial v^i}$ (with v^i the coordinates on the $\mathbb{R}^k$-fiber). Then the horizontal component is $X_H = X^\mu(x, v)\partial_\mu - X^\mu(x, v)A_{\mu j}^{\,i}(x)v^j\frac{\partial}{\partial v^i}$. (Or if you don't like index notations, write $X = X_{\mathrm{base}} + X_{\mathrm{fiber}}$, then $(X_H)_{x,v} = X_{\mathrm{base}}(x, v) - \iota_{X_{\mathrm{base}}(x,v)}A(x)(v)$.) Calculate the Lie bracket $Z = [X_H, Y_H]$ and take the vertical component (with $Z_V = Z - Z_H$). To simplify the computations, you can first check that the expression $[X_H, Y_H]_V$ is $C^\infty(U \times \mathbb{R}^k)$-linear in X and Y (i.e. $[(fX)_H, Y_H]_V = f[X_H, Y_H]_V$ and similarly $[X_H, (fY)_H]_V = f[X_H, Y_H]_V$, for f any function on the total space). This observation allows one to disregard all terms with derivatives of X^μ, Y^μ in the computation of the Lie bracket $[X_H, Y_H]$ (since they will anyway disappear after the subsequent projection to V).

let $A = \sigma^*\mathcal{A} \in \Omega^1(U, \mathfrak{g})$ be the connection 1-form on the base. Show that $\mathcal{A}$ can be expressed in terms of A as

$$\mathcal{A}|_{(x,g)} = g^{-1}dg + g^{-1}A|_x g \tag{3}$$

where $(x, g) \in U \times G$ is a point in the total space.[5]

4. (a) Assume that a connection in a vector bundle $E \to M$ of rank k is described in a local trivialization $\{U_\alpha, \phi_\alpha\}$ by matrix-valued 1-forms $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{gl}(k))$. (I.e. in a trivialization the covariant derivative is $\nabla = d + A_\alpha$.) Show that on an overlap $U_\alpha \cap U_\beta$, one has

$$A_\beta = t_{\beta\alpha}A_\alpha t_{\alpha\beta} + t_{\beta\alpha}dt_{\alpha\beta} \tag{4}$$

with $t_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(k)$ the transition functions.[6]

(b) Assume we have a connection in a principal G-bundle $\mathcal{P} \to M$ determined by a 1-form $\mathcal{A} \in \Omega^1(\mathcal{P}, \mathfrak{g})$. Assume that $\mathcal{P}$ is trivialized over a cover $\{U_\alpha\}$ of M with trivializing local sections $\{\sigma_\alpha \colon U_\alpha \to \mathcal{P}|_{U_\alpha}\}$. Define local 1-forms of the connection as pullbacks $A_\alpha = \sigma_\alpha^*\mathcal{A} \in \Omega^1(U_\alpha, \mathfrak{g})$. Show that on an overlap $U_\alpha \cap U_\beta$ one has again the relation (4) between local connection 1-forms, where now transition functions take values in G.[7] [8]

5. (a) Prove that for any connection in the tautological line bundle τ over $\mathbb{CP}^1$, the integral of the curvature 2-form $F \in \Omega^2(\mathbb{CP}^1)$ is

$$\int_{\mathbb{CP}^1} F = 2\pi i$$

(In particular, there is no flat connection in τ.)[9]

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[5] Hint: using the normalization condition $\iota_{X_\xi}\mathcal{A} = \xi$ show that $\mathcal{A}$ on the unit section must satisfy $\mathcal{A}_{x,1}(\theta, \xi) = \xi + A_x(\theta)$ for $(\theta, \xi) \in T_xU \times T_1G = T_xU \times \mathfrak{g}$ (for that, first show that the fundamental vector field X_ξ arising from the derivative of right G-action on G has the form $(X_\xi)_{x,g} = g\,\xi \in T_gG$; in particular, $(X_\xi)_{x,1} = \xi$). Next, show that G-equivariance allows one to extend $\mathcal{A}$ from the unit section to the entire $U \times G$: $\mathcal{A}_{x,g}(\theta, \underbrace{\xi g}_{=\psi \in T_gG}) = (R_g^*\mathcal{A})_{x,1}(\theta, \xi) = (\mathrm{Ad}_{g^{-1}})_{x,1}(\theta, \xi) = g^{-1}\xi g + g^{-1}A_x(\theta)g = g^{-1}\psi + g^{-1}A_x(\theta)g$, which corresponds to (3).

[6] Hint: use that on $U_\alpha \cap U_\beta$, one has $t_{\beta\alpha}(d + A_\alpha)\sigma_\alpha = (d + A_\beta)\sigma_\beta$ for $\sigma_\beta = t_{\beta\alpha}\sigma_\alpha$, for any local section σ_α ($\mathbb{R}^k$-valued function) over U_α. The equality comes from two ways to write locally $\nabla\sigma$.

[7] Here we assume for simplicity that G is a matrix Lie group. More generally, instead of (4), we should write $A_\beta = \mathrm{Ad}_{t_{\beta\alpha}}A_\alpha + t_{\alpha\beta}^*\mu$ where $\mu \in \Omega^1(G, \mathfrak{g})$ is the Maurer-Cartan left-invariant 1-form on the group (which for a matrix group has the form $\mu = g^{-1}dg$).

[8] Hint: use (3). More explicitly, write the connection 1-form on the total space as $\mathcal{A} = g_\alpha^{-1}dg_\alpha + g_\alpha^{-1}A_\alpha g_\alpha$ in one trivialization chart (at a point $\phi_\alpha(x, g_\alpha) = s_\alpha g_\alpha$, with $(x, g_\alpha) \in U_\alpha \times G$) and as $\mathcal{A} = g_\beta^{-1}dg_\beta + g_\beta^{-1}A_\beta g_\beta$ in the other chart (at the same point in the total space). From $g_\beta = t_{\beta\alpha}(x)g_\alpha$, obtain a relation between A_β and A_α.

[9] Hint: cut $\mathbb{CP}^1$ into two disks $B_\pm$ contained in open sets $D_\pm$ of the trivializing cover for τ from Exercise sheet 1. A connection is represented by local 1-forms $A_\pm$ on $D_\pm$, related on the overlap. Use this to evaluate $\int_{\mathbb{CP}^1} F$ as $\int_{B_+} F + \int_{B_-} F$ where the two integrals can be evaluated in terms of local connection 1-forms $A_\pm$.

(b) Prove that for any connection in $\tau^{\otimes n}$, $n \in \mathbb{Z}$,[10] one has

$$\int_{\mathbb{CP}^1} F = 2\pi i n$$

[10]By convention for any line bundle L, the inverse L^{-1} is understood as the dual bundle L^*. Thus, e.g., $\tau^{\otimes(-5)} = (\tau^*)^{\otimes 5}$. (Generally, isomorphism classes of line bundles over a fixed base M form a group under tensor product, with unit being the trivial line bundle and the inverse given by dualization.)