INTERMEDIATE GEOMETRY AND TOPOLOGY EXERCISES 3, 9/10/2021.

1. **Cellular cohomology of projective spaces.**
   (a) Calculate the cellular cohomology of $\mathbb{RP}^n$ with coefficients in $\mathbb{Z}^2$. Use the standard CW model of $\mathbb{RP}^n$ induced via the covering map $p\colon S^n \to \mathbb{RP}^n$ by the CW decomposition of S^n with two k-cells
$$\begin{aligned} B^k_+ &= \{(x_0, \ldots, x_{k-1}, x_k, 0 \ldots, 0) \in S^n \subset \mathbb{R}^{n+1} \mid x_k > 0\}, \\ B^k_- &= \{(x_0, \ldots, x_{k-1}, x_k, 0 \ldots, 0) \in S^n \subset \mathbb{R}^{n+1} \mid x_k < 0\} \end{aligned}$$
   in each dimension $k = 0, 1, \ldots, n$.
   Show that
$$H^k(\mathbb{RP}^n, \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & 0 \le k \le n \\ 0, & k > n \end{cases}$$
   (as abelian groups).
   (b) (Optional.) Calculate the cup product in $H^\bullet(\mathbb{RP}^2, \mathbb{Z}_2)$. In particular, show that the cup square $a \cup a$ of the generator a of $H^1(\mathbb{RP}^2, \mathbb{Z}_2)$is nonzero.[1]
   (c) Calculate homology and cohomology of $\mathbb{RP}^n$ with coefficients in $\mathbb{Z}$.[2] What does the cup product in $H^\bullet(\mathbb{RP}^n, \mathbb{Z})$ look like?
   (d) (Optional.) Recover the answer of (1a) from the answer of (1c) and the universal coefficient theorem.
   (e) Calculate $H^\bullet(\mathbb{CP}^n, \mathbb{Z})$. Use the standard CW decomposition of $\mathbb{CP}^n$ with a single cell
$$e^{2k} = \{(z_0 : \ldots : z_{k-1} : 1 : 0 : \cdots : 0) \in \mathbb{CP}^n \mid z_0, \ldots, z_{k-1} \in \mathbb{C}\}$$
   in each even dimension $2k$, $k = 0, 1, \ldots, n$.

2. Compute all Stiefel-Whitney numbers for $(\mathbb{RP}^2 \times \mathbb{RP}^2) \sqcup \mathbb{RP}^4$.

[1]One possible route is as follows. Switch to singular homology/cohomology. Use Poincaré duality $H^i(M, \mathbb{Z}_2) \xrightarrow{\sim} H_{\dim M - i}(M, \mathbb{Z}_2)$ to convert the question to computing the intersection in homology $H_\bullet(\mathbb{RP}^2, \mathbb{Z}_2)$. The interesting case is showing that $b \cap b = 1 \in H_0(\mathbb{RP}^2, \mathbb{Z}_2)$ – the homology class of a point, where b is the generator of $H_1(\mathbb{RP}^2, \mathbb{Z}_2)$ Poincaré dual to a, the generator of $H^1(\mathbb{RP}^2, \mathbb{Z}_2)$.

[2]First show that the CW chain complex takes the form $0 \leftarrow \underbrace{\mathbb{Z}}_{C_0} \xleftarrow{0} \underbrace{\mathbb{Z}}_{C_1} \xleftarrow{2\cdot} \underbrace{\mathbb{Z}}_{C_2} \xleftarrow{0} \underbrace{\mathbb{Z}}_{C_3} \xleftarrow{2\cdot} \underbrace{\mathbb{Z}}_{C_4} \xleftarrow{0} \cdots \leftarrow \underbrace{\mathbb{Z}}_{C_n} \leftarrow 0$. I.e. the boundary map alternates between the zero map and multiplication by 2.