INTERMEDIATE GEOMETRY AND TOPOLOGY EXERCISES 5, 9/24/2021.

1. (Chern character) For L a complex line bundle over M,[1] define the *Chern character* as the following element in the rational cohomology of the base

$$\mathrm{ch}(L) := 1 + c_1(L) + \frac{1}{2!}c_1(L)^2 + \frac{1}{3!}c_1(L)^3 + \cdots = e^{c_1(L)} \quad \in H^\bullet(M;\mathbb{Q})$$

Here by Chern classes we understand the image of usual Chern classes (living in $H^\bullet(M;\mathbb{Z})$) in $H^\bullet(M;\mathbb{Q})$ under the inclusion of coefficients $\mathbb{Z} \hookrightarrow \mathbb{Q}$.

For a complex vector bundle over M splitting as a sum of line bundles, $E = L_1 \oplus \cdots \oplus L_k$, set

$$\mathrm{ch}(E) := e^{c_1(L_1)} + \cdots + e^{c_1(L_k)} \quad \in H^\bullet(M;\mathbb{Q})$$

(a) Show that for vector bundles E, E' over M, each splitting as a sum of line bundles, one has[2]

$$\mathrm{ch}(E \oplus E') = \mathrm{ch}(E) + \mathrm{ch}(E') \tag{1}$$
$$\mathrm{ch}(E \otimes E') = \mathrm{ch}(E)\,\mathrm{ch}(E') \tag{2}$$

(Note that this is different from the behavior of the total Chern class which instead satisfies $c(E \oplus E') = c(E)c(E')$ and has no nice compatibility with tensor products in general).

(b) Consider the ring of symmetric polynomials $\mathrm{Sym}_k := \mathbb{Q}[x_1, \ldots, x_k]^{S_k}$ of k variables $x_1, \ldots, x_k$. Elementary symmetric polynomials $\sigma_r = \sum_{1 \leq i_1 < \cdots < i_r \leq k} x_{i_1} \cdots x_{i_r}$ with $r = 1, \ldots, k$ are known to freely generate Sym_k (as a ring over $\mathbb{Q}$). Consider the polynomials $s_p := \sum_{i=1}^k (x_i)^p$ – sum of p-th powers of variables, $p = 1, 2, \ldots$. Show that the first few s_p polynomials have the following expressions in terms of elementary ones:

$$s_1 = \sigma_1, \quad s_2 = \sigma_1^2 - 2\sigma_2, \quad s_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3$$

Generally, s_p is a polynomial $F_p(\sigma_1, \ldots, \sigma_k)$ is σ_r's, with coefficients independent of k - why?

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[1] For this problem we are free to choose between smooth category and topological category. E.g. we can choose the latter and say that M is just a topological space and all relevant vector bundles are in topological category (continuous local trivializations and transition functions).

[2] Recall that for line bundles we know the property $c_1(L \otimes L') = c_1(L) + c_1(L')$.

(c) Show that for a complex vector bundle E splitting as a sum of line bundles, one has[3]

$$\mathrm{ch}(E) = k + c_1(E) + \frac{1}{2!}(c_1(E)^2 - 2c_2(E)) + \\ + \frac{1}{3!}\big(c_1(E)^3 - 3c_1(E)c_2(E) + 3c_3(E)\big) + \cdots \tag{3}$$

where the term in $H^{2i}(M;\mathbb{Q})$ is $F_i(c_1(E),\ldots,c_k(E))$, with polynomials F_p as in (1b)

(d) Show that for E, E' two complex vector bundles of ranks k, k', each splitting as a sum of line bundles, one has

$$c_1(E \otimes E') = k'\ c_1 + k\ c_1', \tag{4}$$

$$c_2(E\otimes E') = \frac{(k')^2 - k'}{2} c_1^2 + \frac{k^2-k}{2}(c_1')^2 + (kk'-1)c_1c_1' + k'c_2 + kc_2' \tag{5}$$

where in the r.h.s. c_i, c_i' stand for the Chern classes of E and E' respectively.

(e) *Splitting principle* asserts that for any (complex or real) vector bundle E over M, one can find a space X and a map $f\colon X \to M$ such that
- The bundle f^*E over X splits as a Whitney sum of line bundles.
- The pullback map in cohomology $f^*\colon H^\bullet(M) \to H^\bullet(X)$ is injective.

Define the Chern character for any complex vector bundle E over M (not necessarily splitting into line bundles) by the formula (3). Using the splitting principle, show that it satisfies (1), (2).[4]
Also, show that expressions (4), (5) for Chern classes of a tensor product hold for E, E' which do not necessarily split into line bundles.

2. Prove that the infinite-dimensional sphere[5] S^∞ is contractible – find an explicit homotopy between the identity map $S^\infty \to S^\infty$ and a constant map $S^\infty \to \mathrm{pt} \in S^\infty$.

[3]Recall that for $E = L_1 \oplus \cdots \oplus L_k$, the total Chern class is, by the multiplicativity property, $c(E) = (1 + c_1(L_1)) \cdots (1 + c_1(L_k)) = 1 + \sigma_1(c_1(L_1),\ldots,c_1(L_k)) + \cdots + \sigma_k(c_1(L_1),\ldots,c_1(L_k))$.

[4]It might be useful as an intermediate step to show that an equivalent definition of $\mathrm{ch}(E)$ is: let $f\colon X \to M$ be the map guaranteed by the splitting principle and let $f^*E = L_1 \oplus \cdots \oplus L_k$ be the corresponding splitting over X. Then $\mathrm{ch}(E) \in H^\bullet(M;\mathbb{Q})$ is uniquely defined (why uniquely?) by $f^*\mathrm{ch}(E) = e^{c_1(L_1)} + \cdots + e^{c_1(L_k)}$.

[5]We define S^∞ as the set of sequences $(x_1, x_2, \ldots) \in \mathbb{R}^\infty$ where only finitely many x_i's can be nonzero (which is our notion of $\mathbb{R}^\infty$) and where $\sum_i x_i^2 = 1$; S^∞ comes with the direct limit topology, $S^\infty = \varinjlim S^n$ under equatorial inclusions $S^n \hookrightarrow S^{n+1}$.