INTERMEDIATE GEOMETRY AND TOPOLOGY EXERCISES 6, 10/1/2021.

1. Prove that a principal G-bundle $\mathcal{P}$ over any compact 3-manifold M for $G = SU(2)$ (or more generally for any compact simply-connected Lie group[1] G) is necessarily a trivial bundle.[2]

   For a principal $SU(2)$-bundle $\mathcal{P}$ over M, for the second Chern class one has the Chern-Weil representative

$$c_2(\mathcal{P}) = \left[\frac{1}{8\pi^2}\mathrm{tr}\,(F_{\mathcal{A}} \wedge F_{\mathcal{A}})\right] \in H^4_{\text{de Rham}}(M) \tag{1}$$

   for $\mathcal{A}$ any connection in $\mathcal{P}$.

2. Consider manifold X of dimension $n \geq 4$, let $\mathcal{P} = X \times SU(2)$ be the trivial $SU(2)$-bundle, and let $\omega = \frac{1}{8\pi^2}\mathrm{tr}(F_{\mathcal{A}} \wedge F_{\mathcal{A}}) \in \Omega^4(X)$ be the Chern-Weil 4-form representing $c_2(\mathcal{P})$ (with $\mathcal{A}$ some connection which due to triviality of $\mathcal{P}$ can be represented by a global 1-form $A \in \Omega^1\big(X, \mathfrak{su}(2)\big)$). Show that ω is exact, $\omega = d\psi$ with

$$\psi = \frac{1}{8\pi^2}\mathrm{tr}\,\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right) \quad \in \Omega^3(X) \tag{2}$$

   (This ψ is called the Chern-Simons 3-form.)

3. Let $\mathcal{P}$ be a principal $SU(2)$-bundle over S^4 defined by the clutching function $t\colon S^3 \to SU(2)$ (where S^3 is the equator of S^4). Show that for the second Chern class one has[3]

$$\langle c_2(\mathcal{P}), [S^4]\rangle = \mathrm{degree}(f)$$

4. Let M be a compact oriented 3-manifold equipped with a trivial $SU(2)$-bundle $\mathcal{P}$. Prove that if we set ${}^gA = gAg^{-1} + gdg^{-1}$ – the gauge transformation of a connection 1-form, then

$$\int_M \psi({}^gA) - \int_M \psi(A) \quad \in \mathbb{Z} \tag{3}$$

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[1] You may use the fact $\pi_2(G) = 0$ for any compact group G.

[2] Hint: recall that for X an n-dimensional CW complex and Y an n-connected topological space (i.e., $\pi_j(Y) = 0$ for $j = 0, \ldots, n$), any continuous map is homotopic to a constant map. To transition to smooth setting, use Whitney approximation theorem (if X, Y are smooth manifolds, then for any continuous map $f\colon X \to Y$ there is a homotopic smooth map $\tilde{f}\colon X \to Y$).

[3] Hint: use (1). Let $D_\pm$ be the top/bottom 3-disks into which S^4 is cut by the equator. Set $A_- = 0$ on D_- and $A_+ = g^{-1}dg$ (the pullback of the Maurer-Cartan 1-form by g) on D_+, where $g\colon D_+ \to G$ is a group-valued function on the disk with boundary restriction $g|_{\partial D_+} = f = t_{-+}$ – the given transition (clutching) function. Check that $A_\pm$ glues into a connection on $\mathcal{P}$. Show that $\int_{S^4} \frac{1}{8\pi^2}\mathrm{tr}\ F \wedge F = \int_{D_+} d\psi(A_+) + \int_{D_-} d\psi(A_-) = \int_{S^3} \psi(f^{-1}df) = \int_{S^3} f^*\Theta$ where $\Theta = -\frac{1}{24\pi^2}\mathrm{tr}\ (h^{-1}dh)^{\wedge 3} \in \Omega^3(SU(2))$ is a volume form on $SU(2)$ satisfying (you don't have to check it) $\int_{SU(2)} \Theta = 1$. Here $h \in SU(2)$ and $h^{-1}dh \in \Omega^1(SU(2), \mathfrak{su}(2))$ the Maurer-Cartan 1-form.

with $\psi(A)$ as in (2).[4]

[4] Idea: use that the oriented cobordism group $\Omega_3 = 0$, i.e. that an oriented closed M is a boundary of some compact oriented 4-manifold $N = N_+$; let N_- be a copy of N with reversed orientation. Let $\mathcal{P}_\pm$ be the trivial $SU(2)$-bundle over $N_\pm$. Let A_+ be a connection on $\mathcal{P}_+$ restricting to A on the boundary $M = \partial N_+$, and let A_- be a connection on $\mathcal{P}_-$ restricting to ${}^gA$ on the boundary M. Show that connections $A_\pm$ can be glued into a connection $\tilde{\mathcal{A}}$ on the $SU(2)$-bundle $\tilde{\mathcal{P}}$ over $\tilde{N} = N_+ \cup_M N_-$ which is trivial over $N_\pm$ and has transition function $t_{-+} = g$ on (a tubular neighborhood of) $M \subset N$. Show that $\langle c_2(\tilde{\mathcal{P}}), [\tilde{N}]\rangle$ on the one hand is an integer and on the other hand is the l.h.s. of (3).