INTERMEDIATE GEOMETRY AND TOPOLOGY EXERCISES 7, 10/8/2021.

1. (Berezin integral.) Let
$$\mathbb{A}_n \quad = \quad \mathbb{C}\langle \theta^1, \ldots, \theta^n \rangle \;/\; \theta^i\theta^j = -\theta^j\theta^i \;\; \forall i,j$$
be the supercommutative algebra generated by anticommuting generators $\theta^1, \ldots, \theta^n$. Show that an element $f \in \mathbb{A}_n$ can be uniquely represented as
$$f = \sum_{k=0}^{n} \sum_{0 \le i_1 < \cdots < i_k \le n} f_{i_1 \cdots i_k} \theta^{i_1} \cdots \theta^{i_k} \tag{1}$$
with $f_{i_1 \cdots i_k} \in \mathbb{C}$ some constants (including $f_\varnothing \in \mathbb{C}$ giving a constant term in f). Define a linear map $\mathsf{B}\colon \mathbb{A}_n \to \mathbb{C}$ (the "Berezin integral") which maps f to the coefficient $f_{12\cdots n}$ of the top monomial $\theta^1 \cdots \theta^n$ in (1). For F an anti-symmetric $n \times n$ matrix with entries F_{ij}, prove that
$$\mathsf{B}(e^{\hat{F}}) = \mathrm{Pf}(F) \tag{2}$$
where $\hat{F} = \sum_{1 \le i < j \le n} F_{ij}\theta^i\theta^j \in \mathbb{A}_n$ is the quadratic polynomial associated to F and $\mathrm{Pf}(F)$ is the Pfaffian.[1]

More abstractly, for V an n-dimensional vector space and a fixed element $\mu \in \wedge^n V$ (a "Berezinian"), one has a map $\mathsf{B}_\mu\colon \wedge^\bullet V^* \to \mathbb{C}$ which maps $f \mapsto \left\langle \mu, f^{(n)} \right\rangle$ where $f^{(n)}$ is the component of f in the top exterior power and $\left\langle , \right\rangle$ is the pairing between $\wedge^n V$ and $\wedge^n V^*$ defined by $\left\langle v_n \wedge \cdots \wedge v_1, \alpha^1 \wedge \cdots \wedge \alpha^n \right\rangle := \det(\langle v_i, \alpha^j \rangle)_{i,j}$ with $\langle v_i, \alpha^j \rangle = \alpha^j(v_i)$ the canonical pairing between V and V^*.

2. (Levi-Civita connection on S^2 in complex coordinates.)
   (a) Show that the Riemannian metric on $\mathbb{CP}^1 = S^2$ induced from the standard metric $\sum_{i=1}^3 (dx^i)^2$ on $\mathbb{R}^3$ via the embedding $S^2 \hookrightarrow \mathbb{R}^3$ as a unit sphere locally has the form[2]
$$g|_{D_+} = \frac{dz \cdot d\bar{z}}{(1+z\bar{z})^2}, \quad g|_{D_-} = \frac{dw \cdot d\bar{w}}{(1+w\bar{w})^2} \tag{3}$$
   Here $D_+ = \{(1:z) | z \in \mathbb{C}\}$, $D_- = \{(w:1) | w \in \mathbb{C}\}$ are the standard complex charts on $S^2 = \mathbb{CP}^1$.

   (b) Levi-Civita connection (the unique torsion-free connection compatible with metric, i.e., with parallel transport preserving the inner product of tangent vectors) in the tangent bundle of a Riemannian manifold M with metric g is

[1] Recall that for n odd, the Pfaffian of an $n \times n$ anti-symmetric matrix is defined to be zero.

[2] Note that $dz \cdot d\bar{z} = \frac{1}{2}(dz \otimes d\bar{z} + d\bar{z} \otimes dz)$ is the commutative product, not the wedge product. E.g., if $z = u + iv$, $\bar{z} = u - iv$ for local real coordinates u, v, then $dz \cdot d\bar{z} = (du)^2 + (dv)^2$.

locally given by the covariant derivative operator acting on sections of TM (vector fields) as $\nabla\colon X \mapsto dx^j\, \frac{\partial X^i}{\partial x^j}\underline{\partial}_i + dx^j\, \Gamma^i_{jk} X^k \underline{\partial}_i$. Here the summation convention over repeated indices is implied, $X = X^i\underline{\partial}_i$, $\underline{\partial}_i = \frac{\partial}{\partial x^i}$ are the basis vectors in T_xM associated to the coordinate chart $\{x^i\}$;[3]

$$\Gamma^i_{jk} = \frac{1}{2}(g^{-1})^{il}(\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk})$$

are the "Christoffel symbols." Here g_{ij} are the components of the metric in local coordinates: $g = g_{ij}dx^i dx^j$; $(g^{-1})^{ij}$ are the entries of the inverse matrix. Calculate the Christoffel symbols on S^2 using coordinates $x^1 = z, x^2 = \bar{z}$ and metric as in (2a). Show that on D_+ one has

$$\Gamma^z_{zz} = -\frac{2\bar{z}}{1+z\bar{z}},\ \Gamma^{\bar{z}}_{\bar{z}\bar{z}} = -\frac{2z}{1+z\bar{z}} \tag{4}$$

while all other components vanish $\Gamma^z_{z\bar{z}} = \Gamma^z_{\bar{z}\bar{z}} = \cdots = 0$. Put another way, the covariant derivative operator $\nabla\colon \Omega^\bullet(S^2, TS^2) \to \Omega^{\bullet+1}(S^2, TS^2)$ is

$$\nabla = dz\frac{\partial}{\partial z} - dz\frac{2\bar{z}}{1+z\bar{z}}\,\underline{\partial}_z \otimes dz + c.c. \tag{5}$$

c.c. stands for "complex conjugate." In D_- one has similar expressions (replacing z with w everywhere).

Write the local expression for the Ehresmann connection 1-form $\mathcal{A} \in \Omega^1(TS^2, \pi^*TS^2)$ as a form on the total space (with $\pi\colon TS^2 \to S^2$ the bundle projection and $\pi^*TS^2 = T^{\mathrm{vert}}(TS^2)$ the vertical tangent bundle of TS^2).

(c) The curvature of the Levi-Civita connection (the Riemann curvature tensor[4]) $R \in \Omega^2(M, \mathrm{End}(TM))$ is locally given in terms of Christoffel symbols as $R = R_{ij}{}^k{}_l (dx^i \wedge dx^j) \otimes (\underline{\partial}_k \otimes dx^l)$ with components

$$R_{ij}{}^k{}_l = \partial_i \Gamma^k_{jl} - \partial_j \Gamma^k_{il} + \Gamma^k_{ir}\Gamma^r_{jl} - \Gamma^k_{jr}\Gamma^r_{il}$$

Calculate the curvature of the Levi-Civita connection on S^2 found in (2b) – show that[5]

$$R = \frac{2dz\wedge d\bar{z}}{(1+z\bar{z})^2}(\underline{\partial}_z \otimes dz - \underline{\partial}_{\bar{z}} \otimes d\bar{z}) \tag{6}$$

(d) (Example of Gauss-Bonnet.) Show that $e_1 = (1+z\bar{z})(\underline{\partial}_z + \underline{\partial}_{\bar{z}})$, $e_2 = i(1+z\bar{z})(\underline{\partial}_z - \underline{\partial}_{\bar{z}})$ is an orthonormal (w.r.t. the metric (3)) real basis in the tangent space to S^2 compatible with the standard orientation of S^2. Show that in terms of this basis, one has, locally on D_+,

$$R = \begin{pmatrix} 0 & R_{12} \\ -R_{12} & 0 \end{pmatrix} \quad \in \Omega^2(D^+) \otimes \mathfrak{so}(2)$$

with

$$R_{12} = \frac{2i\; dz \wedge d\bar{z}}{(1+z\bar{z})^2}$$

[3]The notation convention here is that $\underline{\partial}_i$ with underline is a basis vector in T_xM whereas ∂_i without underline is the differential operator $\frac{\partial}{\partial x^i}$.

[4]A historical convention is to use R rather than F to denote the curvature of Levi-Civita connection.

[5]Another way to get the result is to calculate the square of the differential operator (5).

Show that

$$\frac{1}{2\pi}\int_{S^2} R_{12} = 2$$

– the Euler characteristic of the 2-sphere.

Mathai-Quillen representative of the Thom class

For details, see E. Getzler "The Thom class of Mathai-Quillen and probability theory," `https://cpb-us-e1.wpmucdn.com/sites.northwestern.edu/dist/c/2278/files/2019/08/thom.pdf`

Let $\pi\colon E \to M$ be an oriented real vector bundle of rank n with metric $(,)$ in fibers. *Mathai-Quillen representative* for the Thom class of E is defined as follows. Let $Y = \oplus_{i\geq 0}\Omega^i(E, \wedge^i\pi^*E)$ viewed as a commutative algebra (with the product coming from the wedge product in forms on E and wedge product in coefficients $\wedge^\bullet\pi^*E$). Let $\Xi \in \Gamma(E, \pi^*E)$ be the tautological section $(x, \xi \in E_x) \mapsto \xi$. Choose some connection in E compatible with metric, represented by a 1-form $\mathcal{A} \in \Omega^1(E, \pi^*E)$, with curvature 2-form on the base $F \in \Omega^2(M, \mathfrak{so}(E))$ ($\mathfrak{so}(E)$ is a vector bundle with fiber over x being the space of skew-symmetric endomorphisms of E_x). Identifying $\mathfrak{so}(E) \simeq \wedge^2 E$,[6] we have $F \in \Omega^2(M, \wedge^2 E)$, thus we can construct the element

$$S = \underbrace{-\frac{1}{\epsilon}(\Xi,\Xi)}_{S_0} + \underbrace{\mathcal{A} + \frac{\epsilon}{2}\pi^* F}_{S'} \quad \in Y \tag{7}$$

with $\epsilon > 0$ a fixed number. Consider the differential form

$$\omega \colon = (\pi\epsilon)^{-\frac{n}{2}}\mathsf{B}(e^S) = (\pi\epsilon)^{-\frac{n}{2}} e^{S_0}\mathsf{B}(e^{S'}) \qquad \in \quad \Omega^n(E) \tag{8}$$

where $\mathsf{B}\colon Y \to \Omega^n(E)$ is the Berezin integral in fibers of the coefficient bundle $\wedge^\bullet\pi^*E$ over E, $\quad \mathsf{B}\colon (\wedge^i\pi^*E)_{x,\xi} \to \mathbb{C}$ (vanishing unless $i = n$), corresponding to a canonical Berezinian $\mu_{x,\xi} = v_n \wedge \cdots \wedge v_1$ for $\{v_i\}$ any orthonormal basis in E_x^* compatible with orientation. The form ω turns out to be a Gaussian-shaped Thom form on E, i.e.,

- $\int_{E_x} \omega = 1$,
- $d\omega = 0$,
- Under a change of connection $\mathcal{A}$ (and under a change of ϵ), ω changes by an exact form.

Locally, in a trivialization neighborhood $U \subset M$ for E, one has

$$S = -\frac{1}{\epsilon} g_{ab}\xi^a\xi^b + \theta_a(d\xi^a + A^a{}_b\xi^b) + \frac{\epsilon}{4}\theta_a\theta_b(g^{-1})^{bc}F^a{}_c \tag{9}$$

Here we chose some (possibly not orthonormal) basis e_a in fibers E_x; ξ^a are the corresponding coordinates on the fiber E_x; g_{ab} are the components of the fiber metric; θ_a are generators of the exterior algebra $\wedge^\bullet E_x$; $A^a{}_b \in \Omega^1(U)$ are the components of the local connection 1-form on the base, $A \in \Omega^1(U, \mathrm{End}(E))$; $F^a{}_b \in \Omega^2(U)$ are the components of the local curvature 2-form on the base, $F \in \Omega^2(U, \mathrm{End}(E))$.

Then, the Mathai-Quillen Thom form ω is locally written as

$$\omega = (\pi\epsilon)^{-\frac{n}{2}}\sqrt{\det g}\cdot \mathsf{B}(e^S) \tag{10}$$

[6] We use the following identification between $\mathfrak{so}(V)$ and $\wedge^2 V$ for V a Euclidean vector space: $\Phi \in \mathfrak{so}(V)$ corresponds to $\sum_{i<j}(e_i, \Phi(e_j))e_i \wedge e_j \in \wedge^2 V$ with $\{e_i\}$ any orthonormal basis in V.

where $\mathsf{B}(\cdots)$ is the Berezin integral over θ_a's, i.e., it returns the coefficient of $\theta_1 \cdots \theta_n$; $\det g$ is the determinant of the matrix g_{ab}.[7]

3. (a) Show that if $s_0 \colon M \to E$ is the zero-section of an oriented vector bundle E of rank $n = 2m$ equipped with fiber metric, then the pullback of the Mathai-Quillen Thom form (8), (10) by s_0 yields

$$s_0^* \omega = \frac{1}{(2\pi)^m} \mathrm{Pf}(F) \quad \in \Omega^{2m}(M) \tag{11}$$

– the Chern-Gauss-Bonnet representative of the Euler class.

(b) Show that for the trivial rank 1 bundle $E = M \times \mathbb{R}$, the Thom form (10) becomes the form ω from the problem 3(a) from exercise sheet 4.

(c) Show that the Mathai-Quillen Thom form on TS^2 is given locally on D_+ by (1) from Exercise sheet 4:

$$\omega = \frac{i}{2\pi\epsilon}(1+z\bar{z})^{-2} \left(\left(d\alpha - \frac{2\alpha \bar{z} dz}{1+z\bar{z}} \right) \left(d\bar{\alpha} - \frac{2\bar{\alpha} z d\bar{z}}{1+z\bar{z}} \right) + 2\epsilon dz \wedge d\bar{z} \right) e^{-\frac{\alpha\bar{\alpha}}{\epsilon(1+z\bar{z})^2}} \tag{12}$$

(in the notations of problem 4 from exercise sheet 4).

[7] The appearance of $\sqrt{\det g}$ in (10) accounts for the fact that a basis-independent Berezinian is $\mu = \sqrt{\det g} \cdot D\theta_n \cdots D\theta_1 = \sqrt{\det g} \cdot$ (coordinate Berezinian).