CFT EXERCISES 10, 12/5/2022

1. Classical Wess-Zumino-Witten model

Fix G a connected matrix Lie group G. Recall that the action functional of Wess-Zumino-Witten model on a closed Riemannian surface Σ is given by

$$S_{WZW}(g) = -\frac{i}{4\pi}\int_\Sigma \mathrm{tr}\left(g^{-1}\partial g \wedge g^{-1}\bar{\partial} g\right) - \frac{i}{12\pi}\int_B \mathrm{tr}(\widetilde{g}^{-1}d\widetilde{g})^{\wedge 3}$$

with the field being a map from the surface to the group $g\colon \Sigma \to G$; B is some oriented compact 3-manifold with $\partial B = \Sigma$, $\widetilde{g}$ – some extension of g from Σ into B.

(a) Compute the variation of the action δS w.r.t. variation of the field.
(b) Find the Euler-Lagrange equation of the theory.
(c) Prove that the currents $J = \partial g\, g^{-1}$, $\bar{J} = g^{-1}\bar{\partial} g$ are conserved in the sense that

$$\bar{\partial} J \sim 0 \mod \mathrm{EL}, \qquad \partial \bar{J} \sim 0 \mod \mathrm{EL}$$

(d) Prove that for $\Omega_{1,2}\colon \Sigma \to G_{\mathbb{C}}$ two *holomorphic* maps to the complexified group and $g\colon \Sigma \to G_{\mathbb{C}}$ any smooth map, one has

$$S_{WZW}(\Omega_1 g \overline{\Omega_2}) = S_{WZW}(g)$$

(Hint: use connectedness of G to reduce the problem to infinitesimal deformations of g.)

2. Classical correspondence between 3d Chern-Simons and 2d Wess-Zumino-Witten

Let G again be a matrix Lie group with Lie algebra $\mathfrak{g}$. Let M be a compact oriented 3-manifold with boundary surface Σ (possibly disconnected); assume that Σ is equipped with a complex structure.

Define the Chern-Simons action as

$$S_{CS}(A) = \frac{1}{2\pi}\int_M \mathrm{tr}(\frac{1}{2}A \wedge dA + \frac{1}{3}A \wedge A \wedge A)$$

with the field $A \in \Omega^1(M, \mathfrak{g})$ – a Lie-algebra valued 1-form, which can be thought of as describing a connection in the trivial principal G-bundle on M. Let

$$b(A) = \frac{1}{4\pi}\int_\Sigma \mathrm{tr} A^{1,0} \wedge A^{0,1}$$

be the "boundary term" and let

$$S'_{CS}(A) = S_{CS}(A) + b(A|_\Sigma).$$

(a) Find the variation of $S'_{CS}(A)$ with respect to variation of the field A. Show that the euler-Lagrange equation is the flatness condition $dA + \frac{1}{2}[A, A] = 0$ and find the Noether 1-form on the space of boundary values of the field A.

(b) For $g\colon M\to G$ and $A\in\Omega^1(M,\mathfrak{g})$, let $A^g\colon = g^{-1}Ag+g^{-1}dg$ be the gauge transformation of A. Show that one has

$$S'_{CS}(A^g)-S'_{CS}(A)=iS_{WZW}(g|_\Sigma)+\frac{1}{2\pi}\int_\Sigma \mathrm{tr}A^{1,0}\wedge g^{-1}\bar{\partial}g$$

where the right hand side depends only on the restriction of A and g to Σ.[1]

(c) Show that if the connection A is "pure gauge," i.e., $A=g^{-1}dg$ for some $g\colon M\to G$, then

$$S'_{CS}(A)=iS_{WZW}(g|_\Sigma).$$

[1] I might be off here with normalization factors and right vs. left action of gauge transformations.