CFT EXERCISES 2, 9/5/2022

1. Conformal maps: generalities

(a) Let (M, g), (M', g'), (M'', g'') be three (pseudo-)Riemannian manifolds and let $\phi_1 \colon (M, g) \to (M', g')$, $\phi_2 \colon (M', g') \to (M'', g'')$ be two conformal maps with Ω', Ω'' the respective conformal factors. Show that the composition $\phi_2 \circ \phi_1 \colon (M, g) \to (M'', g'')$ is a conformal map. Find its conformal factor.

(b) Let $\phi \colon (M, g) \to (M', g')$ be a conformal diffeomorphism with conformal factor. Show that the inverse $\phi^{-1} \colon (M', g') \to (M, g)$ is also a conformal diffeomorphism, find its conformal factor.

(c) Let $\phi \colon (M, g) \to (M, g)$ be a conformal map with conformal factor Ω. Show that the same map ϕ regarded as a map $\phi \colon (M, \Lambda \cdot g) \to (M, \Lambda \cdot g)$ is also a conformal map; find its conformal factor. Here Λ is a positive function on M.

2. Examples of conformal maps

(a) **Stereographic projection.** Let $S^n = \{(x^0, \ldots, x^n) | \sum_{i=0}^n (x^i)^2 = 1\}$ be the unit sphere in $\mathbb{R}^{n+1}$ with $N = (1, 0, \ldots, 0)$ the North pole. Consider the map

$$\begin{array}{cccc} \phi\colon & S^n \backslash N & \to & \mathbb{R}^n \\ & (x^0, \ldots, x^n) & \mapsto & \frac{1}{1-x^0}(x^1, \ldots, x^n) \end{array} \tag{1}$$

(the stereographic projection). Consider S^n with standard round metric $g_{S^1} = \sum_{i=0}^n (dx^i)^2$ (the pullback of the standard metric on $\mathbb{R}^{n+1}$ along the inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$) and $\mathbb{R}^n$ (the codomain of (1)) with standard metric $g_{\mathbb{R}^n} = \sum_{i=1}^n (x^i)^2$.

(i) Show that ϕ is a conformal map.
(ii) Find the corresponding conformal factor Ω.

(b) **Inversion.** Show that the inversion map

$$\begin{array}{cccc} \phi\colon & \mathbb{R}^n \backslash \{0\} & \to & \mathbb{R}^n \backslash \{0\} \\ & \vec{x} & \mapsto & \frac{\vec{x}}{||\vec{x}||^2} \end{array}$$

is an involutive orientation-reversing conformal diffeomorphism. Find the conformal factor Ω.

(c) **Holomorphic and antiholomorphic maps.** Let $\phi \colon D \to D'$ be a smooth map between two open sets in $\mathbb{R}^2 = \mathbb{C}$ (equipped with standard metric $g = dx^2 + dy^2 = dz d\bar{z}$ on the source and $g = u^2 + dv^2 = dw d\bar{w}$, where $z = x + iy$, $w = u + iv$ is the complex coordinate on the source and target copy of $\mathbb{C}$).

(i) Show that ϕ is a conformal map if and only if ϕ is either a holomorphic or an antiholomorphic map (do it in real and in complex coordinates, as two independent computations).
(ii) Show that the corresponding conformal factor Ω is $\left|\frac{\partial w}{\partial z}\right|^2$ is ϕ is holomorphic and $\left|\frac{\partial w}{\partial \bar{z}}\right|^2$ is ϕ is anti-holomorphic.

(d) **Möbius transformations.** Show that the group

$$PSL_2(\mathbb{C}) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \middle| \, a, b, c, d \in \mathbb{C}, \; ad - bc = 1 \right\} / \mathbb{Z}_2$$

(where the quotient identifies a matrix with its negative) acts on the Riemann sphere $\bar{\mathbb{C}} = \mathbb{C}P^1$ by "fractional-linear transformations" (or "Möbius transformations")

$$\left(\begin{array}{cc} a & b \\ c & d \end{array} \right) : \quad z \mapsto \frac{az+b}{cz+d} \tag{2}$$

(i) Check that the Möbius transformations (2) are conformal maps. Find the corresponding conformal factor Ω.

(ii) Check that the map $PSL_2(\mathbb{C}) \to \mathrm{Conf}(\mathbb{C}P^1)$ is a group homomorphism: product in the group is mapped to composition of Möbius transformations.