

CFT EXERCISES 5, 9/26/2022

1. EULER-LAGRANGE EQUATIONS IN CLASSICAL FIELD THEORY

Find the Euler-Lagrange equations and the density $\underline{\alpha}$ of the Noether 1-form in the following cases.

- (a) Non-free scalar field on a Riemannian manifold (M, g) :

$$(1) \quad S[\phi] = \int_M \frac{1}{2} d\phi \wedge *d\phi + V(\phi) d\text{vol}_g = \int_M \sqrt{\det g} d^n x \left(\frac{1}{2} (g^{-1})^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right)$$

with $\phi \in C^\infty(M)$ the scalar field, $V(\phi) = \sum_{k=0}^p \frac{a_k}{k!} \phi^k$ a fixed polynomial of degree $p \geq 3$, and $d\text{vol}_g = *1 = \sqrt{\det g} d^n x$ the metric volume form.

- (b) Yang-Mills theory on (M, g) :

$$(2) \quad S[A] = -\frac{1}{2} \int_M \text{tr} F_A \wedge *F_A = -\frac{1}{4} \int \text{tr} F_{\mu\nu} F^{\mu\nu} d\text{vol}_g$$

where the field A is a connection in a fixed principal G -bundle P over M (with G a semi-simple Lie group) and $F_A = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu \in \Omega^2(M, \text{ad}(P))$ is the curvature of the connection; tr is the trace in the adjoint representation of the Lie algebra $\mathfrak{g} = \text{Lie}(G)$.

- (c) Chern-Simons theory on a 3-manifold M :

$$(3) \quad S[A] = \int_M \text{tr} \left(\frac{1}{2} A \wedge dA + \frac{1}{3} A \wedge A \wedge A \right)$$

with the field $A \in \Omega^1(M, \mathfrak{g})$ – the 1-form of a connection in a trivial principal bundle $M \times G \rightarrow M$.

2. STRESS-ENERGY TENSOR

- (a) Find the Hilbert stress-energy tensor¹

$$T^{\mu\nu} = -\frac{2}{\sqrt{\det g}} \frac{\delta S_{M,g}}{\delta g_{\mu\nu}}$$

for the scalar field theory (1). Then check explicitly the conservation property $\nabla_\mu T^{\mu\nu} \sim 0$ modulo Euler-Lagrange equations.

- (b) Find the Hilbert stress-energy tensor for the Yang-Mills theory (2).

¹For evaluating the variation w.r.t. the variation of metric, the following identities are useful: $\delta\sqrt{\det g} = \frac{1}{2} \text{tr}(g^{-1} \delta g) \sqrt{\det g}$ (prove this from $\det g = e^{\text{tr} \log g}$) and $\delta(g^{-1})^{\mu\nu} = -(g^{-1})^{\mu\alpha} \delta g_{\alpha\beta} (g^{-1})^{\beta\nu}$ (or in index-free form: $\delta g^{-1} = -g^{-1} \delta g g^{-1}$ where r.h.s. is understood as a matrix product).

3. BEHAVIOR OF HODGE STAR UNDER WEYL TRANSFORMATIONS

Let $*_g$ be the Hodge star associated to a metric g on an n -dimensional manifold M . Show that, for $g' = \Omega \cdot g$, with Ω a positive function on M , and for α any p -form, one has

$$*_{g'}\alpha = \Omega^{\frac{n}{2}-p} \cdot *_g\alpha$$

Note that this implies, in particular, that Hodge star is Weyl-invariant when action on forms of degree $\frac{n}{2}$ (for n even).

4. EXAMPLE OF A NOETHER CURRENT FOR A MIXED SOURCE-TARGET SYMMETRY

Consider the free massless scalar field on Euclidean \mathbb{R}^n , defined by the action

$$S[\phi] = \int \frac{1}{2} d\phi \wedge *d\phi = \int \frac{1}{2} \partial_\mu \phi \partial_\mu \phi d^n x$$

And consider the mixed source/target transformation – dilation on \mathbb{R}^n accompanied by a rescaling of the field value

$$x \mapsto x' = \beta x \quad , \quad \phi(x) \mapsto \phi'(x') = \beta^{-\frac{n}{2}+1} \phi(x)$$

with $\beta > 0$ the scaling parameter. Or, equivalently,

$$\phi(x) \mapsto \phi'(x) = \beta^{-\frac{n}{2}+1} \phi(\beta^{-1}x)$$

Show that this is a symmetry (maps solutions of the EL equation to solutions). Show that the corresponding infinitesimal symmetry changes the Lagrangian density by a term of form $d\Lambda$ – and find Λ . Finally, find the Noether current corresponding to the symmetry.

5. NOETHER CURRENT FOR SOURCE SYMMETRIES IN ABELIAN CHERN-SIMONS THEORY

Consider abelian Chern-Simons on an oriented 3-manifold M , with fields $A \in \Omega^1(M)$ and action

$$S = \int_M \frac{1}{2} A \wedge dA$$

- Show that any vector field r on M is a source symmetry.
- Find the conserved current $J_r \in \Omega_{\text{loc}}^{2,0}(M \times \text{Fields}_M)$ associated with r by Noether theorem.
- Compare J_r with Show that $J'_r = \iota_{\langle T_{\text{Hilb}}, r \rangle} d\text{vol}_g$ – the conserved current arising from contracting the source symmetry with the Hilbert stress-energy tensor – vanishes.
- Are the currents J_r and J'_r equivalent? I.e., do they satisfy

$$J_r - J'_r \underset{EL}{\sim} d(\dots)$$