CFT EXERCISES 9, 11/14/2022

Null-vectors at level 2

(1) Let $|h\rangle$ be a highest vector of a Virasoro Verma module with central charge c and with $L_0|h\rangle = h|h\rangle$ (for some $c, h \in \mathbb{R}$). Under which condition on c and h there exist two numbers $\alpha, \beta \in \mathbb{R}$ not vanishing simultaneously, such that the vector

$$|\chi\rangle :\, = (\alpha L_{-2} + \beta L_{-1}^2)|h\rangle$$

satisfies $L_n|\chi\rangle = 0$ for all $n \geq 1$? (Such a vector is called a null-vector.)

(2) Show that for $c = \frac{1}{2}$, $h = \frac{1}{16}$,

$$(L_{-2} - \frac{4}{3}L_{-1}^2)|\frac{1}{16}\rangle \qquad (1)$$

is a null-vector.

(3) In the free fermion theory, using the expression for Virasoro generators L_n in terms of Clifford generators $\hat{b}_n$, show that the vector (1) is actually zero (as an element of the space of states, not an element of the Verma module).

bc system

In the bc system, one has field operators

$$\hat{c}(z) = \sum_{n\in\mathbb{Z}} \hat{c}_n z^{-n+1}, \quad \hat{b}(z) = \sum_{n\in\mathbb{Z}} \hat{b}_n z^{-n-2}$$

with the operators $\hat{b}_n, \hat{c}_n$ satisfying the anticommutation relations

$$[\hat{b}_n, \hat{c}_m]_+ = \delta_{n,-m}\mathrm{Id}, \quad [\hat{b}_n, \hat{b}_m]_+ = 0, \quad [\hat{c}_n, \hat{c}_m]_+ = 0 \qquad (2)$$

One understands the operators $\hat{b}_{\geq -1}$, $\hat{c}_{\geq 2}$ as annihilation operators and $\hat{b}_{\leq -2}$, $\hat{c}_{\leq 1}$ as creation operators.[1] The normal ordering puts annihilation operators to the right and creation operators to the left; vacuum vector $|\mathrm{vac}\rangle$ is killed by annihilation operators; applying creation operators to it gives nonzero vectors.

(a) Show that one has

$$\langle b(w)c(z)\rangle :\, = \langle s|\mathcal{R}\hat{b}(w)\hat{c}(z)|\mathrm{vac}\rangle = \frac{1}{w-z}$$

where $|\mathrm{vac}\rangle$ is the vector annihilated by $\hat{b}_{>0}, \hat{c}_{>0}$. Here $\langle s|$ is an element of $\mathcal{H}^*$ which is killed by acting on it on the right with any creation operator.

Also show that one has the OPE

$$\mathcal{R}\hat{b}(w)\hat{c}(z) = \frac{\mathrm{Id}}{w-z} + \mathrm{reg.}$$

[1] Note that this is forced by requiring that the state-field correspondence sends fields c and b to well-defined nonzero vectors in $\mathcal{H}$.

(b) The stress-energy of the system is defined as

$$\hat{T}(z) =: 2\partial\hat{c}(z)\hat{b}(z) + \hat{c}(z)\partial\hat{b}(z) : .$$

Compute (using Wick's lemma) the OPEs $T(w)b(z)$, $T(w)c(z)$ and show that b, c are primary fields, with $h = 2$ and $h = -1$, respectively.

(c) Compute the OPE $T(w)T(z)$ and show that the central charge of the model is $c = -26$.

(d) Consider the modification of the system with $\hat{T}_j(z) =: \partial\hat{c}(z)\hat{b}(z) + j\partial(\hat{c}(z)\hat{b}(z)) :$, with $j \in \mathbb{R}$ a parameter. Find the conformal weights of fields b, c in this theory and find the central charge.

Bosonic string

Consider the CFT consisting of D free scalar fields $\phi_1, \dots, \phi_D$, a bc system and a complex conjugate $\bar{b}\bar{c}$-system. Consider the field

$$J = cT_{\text{bosons}} + \frac{1}{2} : cT_{bc} :=: c\left(-\frac{1}{2}\sum_{k=1}^{D}\partial\phi_k\partial\phi_k\right) + c\partial cb :$$

(a) Find the singular part of the OPE $J(w)J(z)$. Show that it is purely regular if and only if $D = 26$.

(b) Introduce the operator $Q\colon V \to V$, $\Phi(z) \mapsto \frac{-1}{2\pi i}\oint_\gamma dw J(w)\Phi(z)$ where the integration contour γ goes around z. Show that if $D = 26$, then $Q^2 = 0$.

(c) Show that one has $Q(b) = T$, where $T = T_{\text{bosons}} + T_{bc}$ is the stress-energy tensor.