# A TALK ON MATHAI-QUILLEN FORMALISM

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## 1. Toy case: delta-form on zeroes of a function.

**Question:** Given a manifold $M$ and a smooth function $f : M \to \mathbb{R}$, how can one write a (smeared) delta-form supported on $f^{-1}(0)$?
**Answer:** Consider the normalized Gaussian 1-form

$$\tau = \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{1}{2\epsilon}\xi^2} d\xi \tag{1}$$

on $\mathbb{R}$, with $\epsilon > 0$ the dispersion/smearing parameter, and pull it back to $M$ by $f$. Then, we have

$$\delta^{\epsilon}_{f^{-1}(0)} = f^*\tau = \frac{1}{\sqrt{2\pi\epsilon}} e^{-\frac{1}{2\epsilon}f^2} df \quad \in \Omega^1(M) \tag{2}$$

One has the following properties:

- $\delta^\epsilon_{f^{-1}(0)}$ is a closed form and changes by an exact form as $\epsilon$ or $f$ is changed. In particular, the de Rham cohomology class $[\delta^\epsilon_{f^{-1}(0)}]\in H^1_{\text{de Rham}}(M)$ is invariant under such deformations.[1]
- For $\alpha\in\Omega^{n-1}(M)$, then

$$\lim_{\epsilon\to 0}\int_M \delta^\epsilon_{f^{-1}(0)}\wedge\alpha=\int_{f^{-1}(0)}\alpha \tag{3}$$

(assuming that 0 is a regular value of $f$).
- For $\alpha$ closed, the equality in (3) holds without having to take the limit $\epsilon\to 0$, for any value of $\epsilon$.

Importantly, the form (2) can be expressed as a Berezin integral over the odd line

$$\delta^\epsilon_{f^{-1}(0)}=\frac{i}{\sqrt{2\pi\epsilon}}\int_{\mathbb{R}[-1]}D\pi\; e^{-\frac{1}{2\epsilon}f^2-i\pi df} \tag{4}$$

with $\pi$ an odd auxiliary variable of internal degree $-1$; $D\pi$ is the standard Berezinian on the odd line.[2] Furthermore, introducing a second auxiliary variable $p$ (even, of degree 0), we can write (4) as

$$\delta^\epsilon_{f^{-1}(0)}=\frac{i}{2\pi}\int_{\mathbb{R}[-1]\oplus\mathbb{R}}D\pi\,dp\; e^{ipf-i\pi df-\frac{\epsilon}{2}p^2} \tag{5}$$

$$=\frac{i}{2\pi}\int_{T[1]\mathbb{R}[-1]}D\pi\,dp\; e^{(d+p\frac{\partial}{\partial\pi})(i\pi f-\frac{\epsilon}{2}\pi p)} \tag{6}$$

In the last expression one can think of $\mathbb{R}[-1]=V_{\text{aux}}$ as auxiliary space. Then, the integrand is a closed form on $M\times V_{\text{aux}}$ (in fact, exponential of an exact form), with $d+p\frac{\partial}{\partial\pi}$ viewed as de Rham differential on $M\times V_{\text{aux}}$, and the integral (6) can be understood as a pushforward (fiber integral) of a closed form on $M\times V_{\text{aux}}$ to a closed form on $M$. From this viewpoint, it is obvious that changing $\epsilon$ of $f$ changes the form on $M\times V_{\text{aux}}$ by an exact form and hence the pushforward $\delta^\epsilon_{f^{-1}(0)}$ is also changed by an exact form.

## 2. Representative of the Euler class of a vector bundle

Let $E$ be an oriented vector bundle of rank $m$ over a compact manifold $M$, endowed with:

- a fiber metric $g$,
- a connection $\nabla$ compatible with $g$,

[1] For $M$ compact, this cohomology class is necessarily trivial (as shown by considering (2) for functions $f$ and $-f$).

[2] The factor $i$ accompanying the odd variable $\pi$ in (4) looks unnecessary but will be justified in (6) below.

- a section $s : M \to E$.

Consider the function

$$
\begin{aligned}
(7) \quad S_{MQ} = \frac{1}{2\epsilon} g(s,s) + i\langle \pi, \nabla s\rangle - \frac{\epsilon}{2}\langle \pi, F_\nabla(g^{-1}(\pi))\rangle \\
\in \quad C^\infty(q^*E^*[-1]) = \Omega^\bullet(M, \wedge^\bullet E)
\end{aligned}
$$

where

- $q : T[1]M \to M$ is the bundle projection of the tangent bundle,[3]
- $\pi$ is the (odd, degree $-1$) coordinate in the fiber of the graded vector bundle $E^*[-1] \to M$,[4]
- $F_\nabla \in \Omega^2(M, \mathrm{End}(E))$ is the curvature form of the connection $\nabla$,
- $\epsilon > 0$ is the "smearing parameter" (scaling the fiber metric as $g \to \frac{1}{\epsilon} g$).

In local coordinates $x^i$ on $M$, using a basis of local sections $e_a$ of $E$, (7) reads

$$
(8) \quad S_{MQ} = \frac{1}{2\epsilon} g_{ab} s^a s^b + i\pi_a \left( \theta^i \frac{\partial}{\partial x^i} s^a + \theta^i A^a_{ib} s^b \right) - \frac{\epsilon}{4} \theta^i \theta^j g^{bc} F_{ijc}{}^a \pi_a \pi_b
$$

where $\theta^i = dx^i$ are the fiber coordinates in $T[1]M$ and $s^a$, $g_{ab}$, $g^{ab}$, $A^a_{ib}$ and $F_{ijc}{}^a$ are the local components of the section, fiber metric and its inverse, local connection 1-form on the base and the curvature 2-form, respectively.

Next, consider the fiber Berezin integral

$$
(9) \qquad \Xi = \left( \frac{i}{\sqrt{2\pi\epsilon}} \right)^m \int_{\text{fiber of } E^*[-1] \to M} D\pi \; e^{-S_{MQ}} \qquad \in \Omega(M)
$$

Here $D\pi \in \Gamma(M, \wedge^m E^*)$ is the fiber Berezinian (fermionic integration measure) induced by the fiber metric $g$ and the orientation of the fiber.

Note that the form degree of $\Xi$ is $m$. Indeed, $S_{MQ}$ is concentrated in $\oplus_{k=0}^2 \Omega^k(M, \wedge^k E)$. Thus, $e^{-S_{MQ}}$ is concentrated in $\oplus_{k=0}^m \Omega^k(M, \wedge^k E)$ and the Berezin integral (9) selects the $k = m$ term.

**Theorem 2.1.** *(i) Form $\Xi$ is closed.*

*(ii) Changing the data $s, g, \nabla, \epsilon$ changes $\Xi$ by an exact form, $\Xi \mapsto \Xi + d(\cdots)$.*

*(iii) The class of $\Xi$ in de Rham cohomology $H^m(M)$ is the Euler class of the bundle $E \to M$.*

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[3] Note that, at the level of total spaces, one has $q^*E^*[-1] = T[1]M \oplus E^*[-1]$ with $\oplus$ the Whitney sum of bundles over $M$.

[4] Alternatively, one can think of the symbol $\pi$ as standing for the tautological map $E^*[-1] \to E^*$.

We will give a proof of this theorem below, in Sections 3.2, 4.

The differential form $\Xi$ defined by (7), (9) is called the Mathai-Quillen representative of the Euler class of $E \to M$.

*Remark* 2.2. If $M$ is not compact, the class of $\Xi$ in de Rham cohomology of the complex of forms $\Omega_0(M)$ falling off to zero at infinity (this complex is quasi-isomorphic to forms with compact support) can depend on $s$ (or rather its behavior at infinity). Example: $M = \mathbb{R}$ with coordinate $x$, $E$ – a trivial bundle with fiber $\mathbb{R}$ with fiber metric $g = (d\xi)^2$ and connection $\nabla = dx\frac{\partial}{\partial x}$. Then for $s = 1$ one has $\Xi = 0$, while for $s = x$ one has $\Xi = \frac{1}{\sqrt{2\pi\epsilon}}e^{-\frac{x^2}{2\epsilon}}dx$ which has integral 1 on $M$ and therefore is not a coboundary in $\Omega_0(M)$.

2.1. **Limits of $\Xi$ as $\epsilon \to 0$ (localization) and as $\epsilon \to \infty$ (Chern-Weil).** **$\epsilon \to 0$ limit of $\Xi$.** Assuming that graph$(s) \subset E$ intersects the zero-section $M \subset E$ transversally, we have

$$\lim_{\epsilon \to 0} \Xi = \delta_{s^{-1}(0)} \tag{10}$$

– the (distributional) $\delta$-form on the zero-locus of the section, $s^{-1}(0) \subset M$. (The limit is understood in distributional sense.)

In this limit we see that the cohomology class $[\Xi] \in H^m(M)$ is the Poincaré dual[5] of the homology class of the zero-locus of the section $s$, $[s^{-1}(0)] \in H_{n-m}(M)$ (understood as the oriented intersection of graph$(s) \subset E$ with the zero-section $M \subset E$), which is one of the interpretations of the Euler class.

Limit (10) means that for any differential form $\mathcal{O} \in \Omega(M)$, we have

$$\lim_{\epsilon \to 0} \int_M \Xi \wedge \mathcal{O} = \int_{s^{-1}(0)} \mathcal{O} \tag{11}$$

And if $\mathcal{O}$ is a closed form, then the l.h.s. is independent of $\epsilon$ and reduced to an integral over the zero-locus of $s$. This is a prototypical localization statement. In QFT examples, $M$ is the space of fields, $s(x) = 0$ is the equation of motion and $\mathcal{O}$ is an observable (or a product of observables).

**$\epsilon \to \infty$ limit.**

$$\lim_{\epsilon \to \infty} \Xi = (2\pi)^{-\frac{m}{2}} \mathrm{Pf}(F_\nabla) \tag{12}$$

– the Pfaffian of the curvature of the connection in $E$. This is a Chern-Weil representative for the Euler class of $E$.

[5]Here we need to assume that $M$ is oriented.

2.2. **Example: Euler characteristic.** Let $E = TM$ the tangent bundle of an oriented compact $n$-manifold $M$, section $s$ – a generic vector field on $M$ (i.e. such that its graph in $TM$ intersects the zero-section transversally). Assuming that $M$ is Riemannian, we have a fiber metric in $TM$ and a compatible connection $\nabla_{\mathrm{LC}}$ – the Levi-Civita connection with curvature $R = F_{\nabla_{\mathrm{LC}}} \in \Omega^2(M, \mathrm{End}(TM))$ – the Riemann tensor. Here the integral of (10) over $M$ becomes

$$\lim_{\epsilon\to 0}\int_M \Xi = \sum_{\text{zeroes } x_i \text{ of } s} \underbrace{\pm 1}_{\mathrm{ind}_{x_i} s} \tag{13}$$

– the sum of indices of zeroes of the vectors field (i.e. the oriented intersection number of graph($s$) and the zero-section of $TM$). By Poincaré-Hopf theorem, this is the Euler characteristic of $M$.

On the other hand, the integral of (12) over $M$ becomes

$$\lim_{\epsilon\to\infty}\int_M \Xi = (2\pi)^{-\frac{n}{2}}\int_M \mathrm{Pf}(R) \tag{14}$$

which is the Chern-Gauss-Bonnet formula for the Euler characteristic of $M$.

Thus, Mathai-Quillen representative for the Euler class of $TM$ (at finite $\epsilon$) "interpolates" between the Poincaré-Hopf representative $\sum_i \pm\delta_{x_i}$ and the Chern-Gauss-Bonnet representative $(2\pi)^{-\frac{n}{2}}\mathrm{Pf}(R)$.

## 3. "First-order formalism" for the Mathai-Quillen representative

The integrand in the formula (9) for the Mathai-Quillen representative is the exponential of a quadratic expression in the section $s$. In field theory applications, $s$ contains a derivative and (9) becomes a path integral in the "second-order formalism" (i.e. with the action functional of second order in derivatives). It is useful to rewrite $\Xi$ in "first-order formalism" – as a fiber integral of the exponential of an action linear in $s$. There are two natural ways to do it – "version 1" below is known in the literature on supersymmetry,[6] whereas "version 2" gives a more clear geometric picture in terms of fiber integrals of closed forms.

3.1. **Version 1.** Consider the function

$$\widetilde{S}_{MQ} = \frac{\epsilon}{2} g^{-1}(p,p) - i\langle p, s\rangle + i\langle \pi, \nabla s\rangle - \frac{\epsilon}{2}\left\langle \pi, F_\nabla(g^{-1}(\pi))\right\rangle \in C^\infty(q^*(E^*[-1]\oplus E^*)) \tag{15}$$

[6] See, e.g., (4.2) in [3] and (13), (14) in [1] (the case of A-model).

where $\pi$ is the odd, degree $-1$ fiber coordinate on $E^*[-1]$ as before and $p$ is the even, degree 0 fiber coordinate on $E^*$. Evaluating the Gaussian integral over $p$, we have

$$\left(\frac{i}{2\pi}\right)^m \int_{\text{fiber of } E^*} dp\; e^{-\widetilde{S}_{MQ}} = \left(\frac{i}{\sqrt{2\pi\epsilon}}\right)^m e^{-S_{MQ}} \tag{16}$$

with $S_{MQ}$ as in (7). Here $dp$ is the volume form in the fiber of $E^*$ determined by fiber metric $g$. Thus, we have the following integral presentation for the Mathai-Quillen form $\Xi$ representing the Euler class:

$$\Xi = \left(\frac{i}{2\pi}\right)^m \int_{\text{fiber of } E^*[-1]\oplus E^*} D\pi\; dp\;\; e^{-\widetilde{S}_{MQ}} \tag{17}$$

In the total fiber integration measure $D\pi\; dp$, the dependence on fiber metric $g$ cancels out: it only depends on the orientation of the fiber.

3.2. **Version 2 (pushforward of the exponential of an exact form).** Another version of the construction is as follows: consider the following exact differential form on the total space of $E^*[-1]$:

$$\widehat{S}_{MQ} = d_{E^*[-1]}\Big(-i\langle \pi, s\rangle + \frac{\epsilon}{2} g^{-1}(\pi, \mathcal{A}^*)\Big) \tag{18}$$

$$= \frac{\epsilon}{2} g^{-1}(\mathcal{A}^*, \mathcal{A}^*) - i\langle \mathcal{A}^*, s\rangle + i\langle \pi, \nabla s\rangle - \frac{\epsilon}{2}\langle \pi, F_\nabla(g^{-1}(\pi))\rangle \tag{19}$$

where $\mathcal{A}^* \in \Omega^1(E^*[-1], T^{\mathrm{vert}}E^*)$ is the connection 1-form on the total space of $E^*[-1]$ corresponding to the connection $\nabla^*$ on $E^*[-1]$ – the dual of the connection $\nabla$ on $E$. In a local trivialization, over a trivializing neighborhood $U \subset M$, one has $\mathcal{A}^* = d\pi - A^T\pi$, where $A \in \Omega^1(U, \mathrm{End}(E))$ is the local connection 1-form of $\nabla$.

Then, we have

$$\Xi = \left(\frac{i}{2\pi}\right)^m \int_{\text{fiber of } E^*[-1]\to M} e^{-\widehat{S}_{MQ}} \tag{20}$$

where the expression on the right is understood as a fiber integral of a differential form on the total space of $E^*[-1]$ to a differential form on the base.

Equivalently, (20) can be thought of as a fiber integral of a function on $T[1]E^*[-1]$ to a function on $T[1]M$:

$$\Xi = \left(\frac{i}{2\pi}\right)^m \int_{\text{fiber of } T[1]E^*[-1]\to T[1]M} D\pi\; dP\;\; e^{-\widehat{S}_{MQ}} \tag{21}$$

where $P = d\pi$ viewed as a coordinate in $E^*$-fibers of the vector bundle $T[1]E^*[-1] \to q^*E^*[-1]$.

*Proof of (i) and (ii) of Theorem 2.1.* Differential form (18) on $E^*[-1]$ is exact, thus $e^{-\widehat{S}_{MQ}}$ is closed and changes by an exact form under a deformation of the data $s, g, \nabla, \epsilon$. Therefore, Stokes' theorem for fiber integrals implies

$$d_M \Xi = \mathrm{const} \cdot \int d_{E^*[-1]} e^{\widehat{S}_{MQ}} = 0$$

and for the variation under the change of data one has

$$\delta \Xi = \int d_{E^*[-1]}(\cdots) = d_M(\cdots)$$

Here $\int$ stands for the fiber integral for the vector bundle $E^*[-1] \to M$. □

3.3. **Limit $\epsilon \to 0$.** In the limit $\epsilon \to 0$, the first-order action (18) becomes

$$\widehat{S}_{MQ} = d_{E^*[-1]}\big(-i\langle \pi, s\rangle\big) = -i\langle P, s\rangle + i\langle \pi, ds\rangle \tag{22}$$

and (21) becomes and integral representation of the delta-form on $s^{-1}(0) \subset M$:

$$\delta_{s^{-1}(0)} = \left(\frac{i}{2\pi}\right)^m \int D\pi\, dP \;\; e^{i\langle P,s\rangle - i\langle \pi, ds\rangle} \tag{23}$$

*Remark* 3.1. The expression $\langle \pi, s\rangle$ appearing in (22) (the "gauge-fixing fermion") corresponds by the odd Fourier transform to the Koszul differential $d^{\mathrm{Koszul}} = \langle s, \frac{\partial}{\partial \psi}\rangle$ acting on $C^\infty(E[1]) = \Gamma(M, \wedge E^*)$ (with $\psi$ the odd fiber coordinate on $E[1]$), for which the zeroth homology is $H_0^{\mathrm{Koszul}} = C^\infty(s^{-1}(0))$. Here by the odd Fourier transform we mean the mapping

$$\begin{array}{ccc} C^\infty(E[1]) & \to & C^\infty(E^*[-1]) \\ f(x,\psi) & \mapsto & \widetilde{f}(x,\pi) = \displaystyle\int_{\text{fiber of } E[1]\to M} D\psi\, e^{\langle \psi, \pi\rangle} f(x,\psi) \end{array}$$

3.4. **Comparison of versions 1 and 2.** Note that a choice of connection $\nabla$ on $E$ determines a diffeomorphism of graded supermanifolds[7]

$$\begin{array}{cccc} \Psi_\nabla & : & q^*(E^*[-1] \oplus E^*) & \to & T[1]E^*[-1] \\ & & (x, \theta, \pi, p) & \mapsto & (x, \theta, \pi, P = p + A^T \pi) \end{array} \tag{24}$$

[7]Recall that an Ehresmann connection on a vector bundle $r\colon H \to M$ is a splitting of the tangent bundle $TH \simeq r^*TM \oplus T^{\mathrm{vert}}H$. At the level of total spaces, this yields a diffeomorphism $TH \simeq TM \oplus H \oplus H$. Setting $H = E^*$ and incorporating the degree shifts, one has (24).

Here $x \in M$ and $\theta = dx$ is the fiber coordinate in $T[1]M \to M$. This diffeomorphism identifies the two first-order actions (15) and (18):

$$\widetilde{S}_{MQ} = \Psi^*_\nabla(\widehat{S}_{MQ})$$

In particular, expression (18) for $\widehat{S}_{MQ}$ translates into the following for $\widetilde{S}_{MQ}$:

$$\widetilde{S}_{MQ} = Q\Big( -i\langle \pi, s\rangle + \frac{\epsilon}{2} g^{-1}(\pi, p)\Big) \tag{25}$$

where $Q$ is a cohomological vector field on $q^*(E^*[-1] \oplus E^*)$ arising as a pullback of the de Rham vector field $d_{E^*[-1]}$ on $T[1]E^*[-1]$ by $\Psi_\nabla$. In local coordinates, one has:

$$Q: \qquad x \mapsto \theta, \quad \theta \mapsto 0, \quad \pi \mapsto p + A^T\pi, \quad p \mapsto -F^T_\nabla \pi + A^T p$$

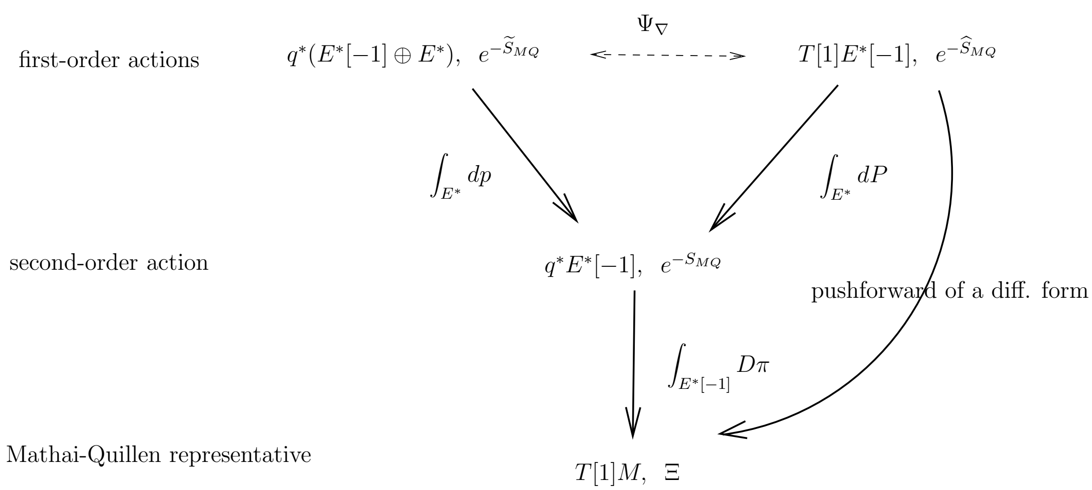



FIGURE 1. A diagram of pushforwards. Top-down arrows are fiber integrals.

## 4. Representative of the Thom class

Consider the function

$$S^{\text{Thom}}_{MQ} = \frac{1}{2\epsilon} g(\xi,\xi) + i\langle \pi, \mathcal{A}\rangle - \frac{\epsilon}{2}\langle \pi, F_\nabla(g^{-1}(\pi))\rangle \tag{26}$$
$$\in C^\infty(u^*E^*[-1]) = \Omega(E, \wedge(r^*E))$$

where

- $\xi$ is the fiber coordinate in $E$,
- $r\colon E \to M$ is the bundle projection and $u\colon T[1]E \to M$ is the composition of bundle projections $T[1]E \to E$ and $E \to M$.

- $\mathcal{A} \in \Omega^1(E, T^{\mathrm{vert}}E)$ is the 1-form of the connection $\nabla$ (locally, $\mathcal{A} = d\xi + A\xi$).

Next, consider the fiber integral

$$\tau = \left(\frac{i}{\sqrt{2\pi\epsilon}}\right)^m \int_{\text{fiber of } u^*E^*[-1]\to T[1]E} D\pi \; e^{-S^{\mathrm{Thom}}_{MQ}} \qquad \in \Omega(E) \tag{27}$$

This is an $m$-form on $E$ depending on $g, \nabla, \epsilon$ (note that the section $s$ is not involved in this construction) with the following properties:

(a) $\tau$ is closed and changes by an exact form under a change of $g, \nabla, \epsilon$.
(b) $\tau$ has Gaussian shape along the fiber of $E$ (in particular, is it fast-decaying as $||\xi|| \to +\infty$).
(c) $\tau$ has integral 1 over each fiber of $E \to M$. Thus, $\tau$ represents the Thom class of $E$ in $H^m_{RD}(E)$.[8]

One can furthermore rewrite (27) in the first-order formalism, as a pushforward of an exact differential form on $E \oplus E^*[-1]$ to $E$:

$$\tau = \left(\frac{i}{2\pi}\right)^m \int_{\text{fiber of } E\oplus E^*[-1]\to E} e^{-\widehat{S}^{\mathrm{Thom}}_{MQ}} \tag{28}$$

where

$$\widehat{S}^{\mathrm{Thom}}_{MQ} = d_{E\oplus E^*[-1]}\Big( -i\langle \pi, \xi\rangle + \frac{\epsilon}{2} g^{-1}(\pi, \mathcal{A}^*)\Big) \tag{29}$$

From this description, the property (a) of $\tau$ above is obvious, using Stokes' theorem for fiber integrals.

*Proof of (iii) of Theorem 2.1.* Comparing differential forms (27) and (9), we see that they are related by the pullback by the section $s$:

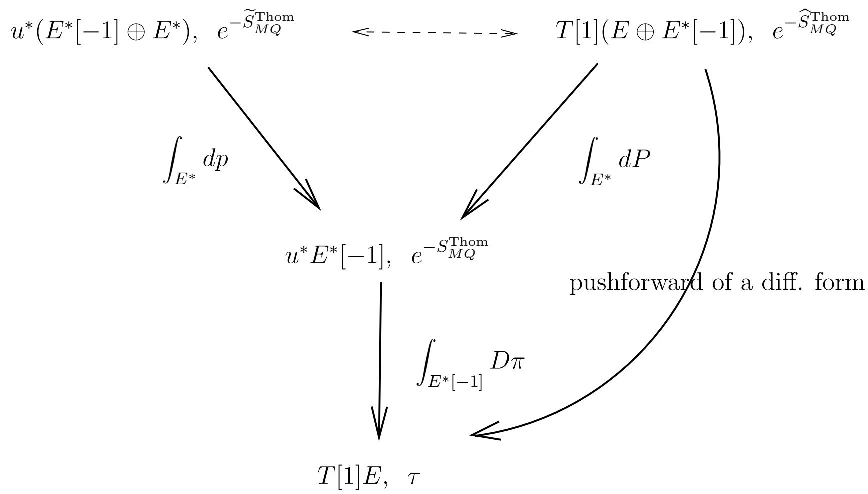
$$\Xi = s^*\tau$$

Knowing that the cohomology class $[\tau] \in H^m_{RD}(E)$ is the Thom class of $E$, we get that its pullback by the section, $[\Xi] = s^*[\tau] \in H^m(M)$, is the Euler class of $E$. □

One can also restate the diagram of pushforwards above for the Thom class instead of Euler class.

The actions $S^{\mathrm{Thom}}_{MQ}$, $\widetilde{S}^{\mathrm{Thom}}_{MQ}$, $\widehat{S}^{\mathrm{Thom}}_{MQ}$ are obtained from their Euler class counterparts by replacing the section $s$ in all the formulae by the fiber coordinate $\xi$ in $E \to M$.

[8]The subscript $RD$ stands for "rapid decay in fiber direction." The cohomology of RD-forms is isomorphic to the cohomology of forms with compact support in the fiber and, in turn, to the cohomology of the Thom space of $E$.




## References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