Introduction to linear algebra and differential equations.

MATH 20580 - 05
9:25 - 10:15
DBRT 136
TA: Shih-Kai Chiu

MATH 20580 - 03
11:30 - 12:20
HH 127
TA: Adrian Pacurar

Organizational:
- [distribute the handout] / web page link: www3.nd.edu/~dyer/Teaching/math20580_S18.html
- 01/22 no class
- first HW due 01/25 to the TA.
- grade = tutorial quizzes + homeworks + MT1 + MT2 + MT3 + final
  max: 550 = 40 + 60 + 100 + 100 + 100 + 150
- my office hours: Wed 4-6 pm, HH124

1.1. Systems of linear equations
Lay, Lay, McDonald

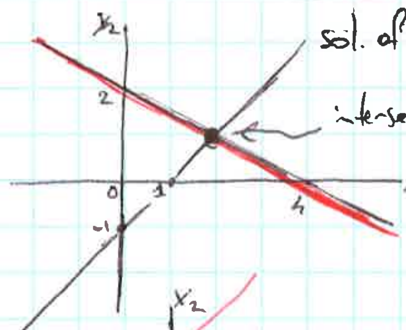
linear equation: $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$

variables (indeterminates): $x_1, \dots, x_n$

coefficients: $a_1, a_2, \dots, a_n$ ⟵ given real/complex numbers: b

System of linear equations ⟶ (set of solutions)
m equations, n variables

Ex: (a) $x_1 - x_2 = 1$ (eq$_1$)
$x_1 + 2x_2 = 4$ (eq$_2$)



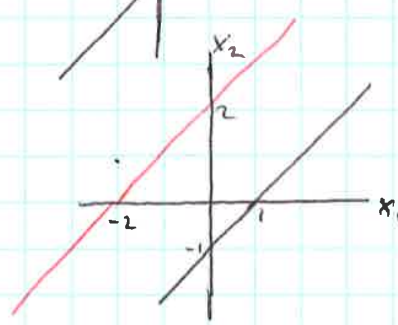
sol. of eq$_1$

intersection - sol. of the system

sol of eq$_2$

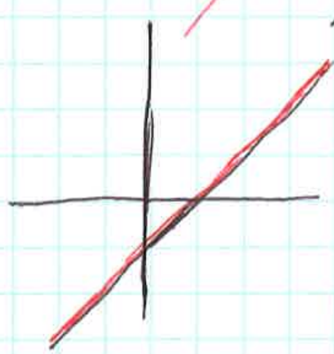
~~$x_1=2, x_2$~~
$(x_1, x_2) = (2, 1)$ - unique solution

(b) $x_1 - x_2 = 1$
$-2x_1 + 2x_2 = 4$



parallel lines → (no solutions)
⇒ system inconsistent

(c) $x_1 - x_2 = 1$
$-2x_1 + 2x_2 = -2$



lines coincide
→ (infinitely many solutions) $(x_1 = x_2 + 1, \ x_2 \text{ any})$

any system of lin. eq. either has ① no solutions] system inconsistent
or ② exactly one solution
or ③ infinitely many solutions] system consistent

Matrix notation

Linear system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 3x_2 - 3x_3 &= 6 \\ 2x_1 + 3x_3 &= 3 \end{aligned}$$

(coefficients of each variable aligned in columns)

→ matrix of coefficients $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & -3 \\ 2 & 0 & 3 \end{bmatrix}$ 3 rows (3 equations), 3 columns (3 variables) } a matrix of size 3×3

→ augmented matrix $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 3 & -3 & 6 \\ 2 & 0 & 3 & 3 \end{array}\right]$ · a 3×4 matrix

← added column of right hand sides

Solving a linear system

Idea: use x_1 term in eq_1 to eliminate x_1 terms in other equations. Then use x_2 term in eq_2 to eliminate x_2 term in other equations etc. ⇝ obtain a very simple equivalent linear system. (i.e. with same solution set)

Ex:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 3x_2 - 3x_3 &= 6 \\ 2x_1 + 3x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 3 & -3 & 6 \\ 2 & 0 & 3 & 3 \end{bmatrix}$$

- keep x_1 in eq_1 and eliminate it from other eq. To do so, add $(-2)\times eq_1$ to eq_3 $\quad r_3 \mapsto r_3 - 2r_1$

$$\begin{array}{l|r} -2\cdot eq_1 & -2x_1 + 4x_2 - 2x_3 = 0 \\ +\ eq_3 & 2x_1 \qquad + 3x_3 = 3 \\ \hline \text{new } eq_3 & 4x_2 + x_3 = 3 \end{array}$$

new system: $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 3 & -3 & 6 \\ 0 & 4 & 1 & 3 \end{bmatrix}$ $\qquad \begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 3x_2 - 3x_3 &= 6 \\ 4x_2 + x_3 &= 3 \end{aligned}$

- multiply eq_2 by $\frac{1}{3}$, to obtain 1 as coeff. of x_2 in eq_2 (optional: simplifies arithmetic in the next step)

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - x_3 &= 2 \\ 4x_2 + x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 4 & 1 & 3 \end{bmatrix} \quad r_2 \mapsto r_2 \cdot \tfrac{1}{3}$$

- use x_2 in eq_2 to eliminate x_2 term in eq_3: replace eq_3 with $eq_3 - 4\cdot eq_2$ $\quad r_3 \mapsto r_3 - 4r_2$

$$\begin{array}{l|r} (-4)\cdot eq_2 & -4x_2 + 4x_3 = -8 \\ +\ eq_3 & 4x_2 + x_3 = 3 \\ \hline \text{new } eq_3 & 5x_3 = -5 \end{array} \qquad \begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - x_3 &= 2 \\ 5x_3 &= -5 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 5 & -5 \end{bmatrix}$$

$r_3 \mapsto r_3 \cdot \frac{1}{5}$

- multiply eq_3 by $\frac{1}{5}$

a system in "triangular" (or "echelon") form →

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - x_3 &= 2 \\ x_3 &= -1 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

- eliminate x_3 from eq_1, eq_2 : $eq_2 \mapsto eq_2 + eq_3$
  $eq_1 \mapsto eq_1 - eq_3$

$$\begin{aligned} x_1 - 2x_2 \quad &= 1 \\ x_2 \quad &= 1 \\ x_3 &= -1 \end{aligned}$$

eliminated the column above x_3

$r_2 \mapsto r_2 + r_3$
$r_1 \mapsto r_1 - r_3$

$$\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$r_1 \mapsto r_1 + 2r_2$

- eliminate x_2 from eq_1 ; $eq_1 \mapsto eq_1 + 2eq_2$

$$\begin{aligned} x_1 \quad &= 3 \\ x_2 \quad &= 1 \\ x_3 &= -1 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

→ we proved that the only solution of original sys. is $(3, 1, -1)$

check: (substitute into orig. sys.)

$$\begin{aligned} 3 - 2\cdot 1 + (-1) &= 0 \\ 1 - 3(-1) &= 4 \\ 2\cdot 3 \quad + 3(-1) &= 3 \end{aligned} \quad \checkmark$$

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solving a lin. sys., we use the operations:

1. replace an equation with itself plus a multiple of another eq.
2. interchange two equations
3. multiply all terms in an equation by a nonzero constant

for the augmented matrix, we perform corresponding elementary row operations

1. (replacement) replace a row with a sum of itself and a multiple of another row
2. (interchange) interchange two rows
3. (scaling) multiply all entries in a row by a nonzero constant.

Two matrices are row equivalent iff they can be transformed into one another by a sequence of elem. row op.

- row operations are reversible
- If the augm. matrices of two lin. sys. are row equivalent then the two systems have same solution set

1.2 Row reduction and echelon forms

17/01/2018 (5)

- leading entry in a row = leftmost nonvanishing entry
- a rectangular matrix is in row echelon form (REF) if
  - all nonzero rows are above zero rows
  - each leading entry in a row is in a column to the right of the row above it
  - all entries in a column below a leading entry are 0

pivot columns, pivot pos.

Ex:
$$\begin{bmatrix} \bullet & * & * & * \\ 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \bullet & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \bullet & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \bullet & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \bullet & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bullet & * \end{bmatrix}$$

$$\begin{bmatrix} 0 & \bullet & * & * & * & * & * \\ 0 & 0 & 0 & \bullet & * & * & * \\ 0 & 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \bullet \end{bmatrix}$$

$\bullet$ $\neq 0$ leading entries / $*$ - any entries

- a matrix is in reduced row echelon form (RREF) if, additionally,
  - all leading entries are 1
  - each leading 1 is the only nonzero entry in its column.

Ex:
$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 1 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- Any matrix A can be row reduced (transformed by a sequence of row operations) into more than one matrix in REF, however the RREF of a matrix is unique.
- Leading ~~matrix~~ entries are always in same positions for any REF of A = pivot positions. A pivot column of A is a column containing a pivot position

leading entries in a REF of A = pivots (numbers)

Row reduction algorithm

$$\text{matrix } A \xrightarrow[\text{"forward phase"}]{\text{steps I - IV}} \text{REF of } A \xrightarrow[\text{"backward phase"}]{\text{step V}} \text{RREF of } A$$

Ex: $A = \begin{bmatrix} 0 & 2 & -6 & -1 & -2 \\ 2 & 1 & 9 & 9 & 6 \\ 2 & 4 & 0 & 6 & 0 \end{bmatrix}$

Step I: begin with leftmost nonzero column: it is a pivot column; pivot pos. is at the top

Step II: select a nonzero entry in pivot col. as the pivot. If necessary, interchange rows to move this entry into pivot pos.

interchange $r_1 \rightleftarrows r_3$

pivot

$$\begin{bmatrix} 2 & 4 & 0 & 6 & 0 \\ 2 & 1 & 9 & 9 & 6 \\ 0 & 2 & -6 & -1 & -2 \end{bmatrix}$$

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Step III Use row replacement to create zeros in all positions below the pivot

$$\xrightarrow[r_2 \mapsto r_2 - r_1]{} \begin{bmatrix} 2 & 4 & 0 & 6 & 0 \\ 0 & -3 & 9 & 3 & 6 \\ 0 & 2 & -6 & -1 & -2 \end{bmatrix}$$

Step IV Cover (or ignore) the row containing the pivot position and all rows above it.
Apply Steps I–III to the remaining submatrix. Repeat until there are no nonzero rows to modify.

$$\begin{bmatrix} \cancel{2} & \cancel{4} & \cancel{0} & \cancel{6} & \cancel{0} \\ 0 & -3 & 9 & 3 & 6 \\ 0 & 2 & -6 & -1 & -2 \end{bmatrix} \xrightarrow[r_3 \mapsto r_3 + \frac{2}{3} r_2]{} \begin{bmatrix} \cancel{2} & \cancel{4} & \cancel{0} & \cancel{6} & \cancel{0} \\ 0 & -3 & 9 & 3 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[r_2 \mapsto -\frac{1}{3} r_2]{\text{(optional)}} \begin{bmatrix} \cancel{2} & \cancel{4} & \cancel{0} & \cancel{6} & \cancel{0} \\ \cancel{0} & \cancel{1} & \cancel{-3} & \cancel{-1} & \cancel{-2} \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

new pivot

REF!

new pivot

already in REF ⇒ IV stops.

If we want RREF:

Step V • beginning with rightmost pivot and working upward and to the left, create zeros above each pivot. If pivot is not 1, make it 1 by rescaling rows.

$$\begin{bmatrix} 2 & 4 & 0 & 6 & 0 \\ 0 & 1 & -3 & -1 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[r_2 \mapsto r_2 + r_3]{r_1 \mapsto r_1 - 6r_3} \begin{bmatrix} 2 & 4 & 0 & 0 & -12 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_1 \mapsto r_1 - 4r_2} \begin{bmatrix} 2 & 0 & 12 & 0 & -12 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \longrightarrow$$

created zeros above pivots

$$\xrightarrow[r_1 \mapsto r_1 \cdot \frac{1}{2}]{\text{rescale}} \begin{bmatrix} 1 & 0 & 6 & 0 & -6 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

← RREF of orig. matrix

Solutions of lin. sys

Suppose augm. mat. of a sys. has been reduced into RREF

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

i.e. system

$x_1 \quad + 3x_3 = -1$

$x_2 + 2x_3 = 5$

$0 = 0$

variables x_1, x_2 corresp. to pivot columns are basic variables,
var. x_3 corresp. to non-pivot col. is a "free variable"

can solve for basic var. in terms of free variables:

$$\begin{cases} x_1 = -1 - 3x_3 \\ x_2 = 5 - 2x_3 \\ x_3 \text{ is free} \end{cases}$$

(takes any value)

- description of all solutions of the lin. sys.

e.g.: can take $x_3 = 1$
⇒ $(-1-3, 5-2, 1)$ is a solution.

• A system is consistent iff REF of Aug. Mat. does not have a row of form $[0 \cdots 0 \; b]$, $b \neq 0$ ⇔ $0 = b$ contradictory eq.
i.e. iff the last column is not pivotal.

• solution of a consistent sys. is unique iff there are no free variables, i.e. no non-pivot columns (except last one)