Vectors in $\mathbb{R}^n$

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$\mathbb{R}^n$ = set of vectors of form $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ ($n\times1$ - matrices)

- can add two vectors in $\mathbb{R}^n$ $\vec u + \vec v$ (vectors must be of same size!!)
- can form a scalar multiple $c \cdot \vec u$

* one has commutativity, associativity, distributivity, like for numbers

- zero vector $\vec 0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\vec 0 + \vec u = \vec u$
- negative of a vector: $-\vec u := (-1)\vec u = \begin{bmatrix} -u_1 \\ -u_2 \\ \vdots \\ -u_n \end{bmatrix}$ then $(-\vec u) + \vec u = \vec 0$
- also, notation: $\vec u + (-1)\vec v =: \vec u - \vec v$.

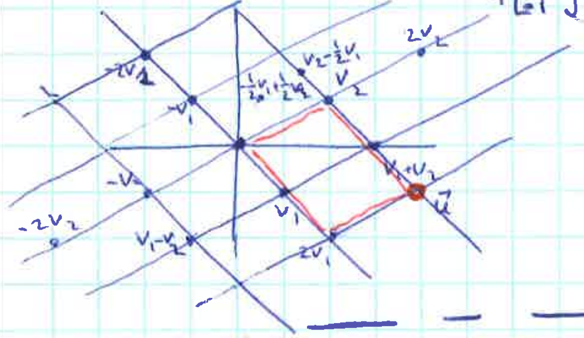
Linear combinations

for vectors $v_1, \dots, v_p \in \mathbb{R}^n$, scalars $c_1, \dots, c_p$, vector

$\boxed{\vec y = c_1 \vec v_1 + \dots + c_p \vec v_p}$ is called a linear combination of $\vec v_1, \dots, \vec v_p$ with weights $c_1, \dots, c_p$

Ex: some linear comb of v_1, v_2 : $3v_1 - \frac{5}{7}v_2$, $\frac{1}{2}v_1 = \frac{1}{2}v_1 + 0 \cdot v_2$, $\vec 0 = 0 \cdot \vec v_1 + 0 \cdot \vec v_2$

Ex: some linear combinations of $v_1 = \begin{bmatrix} +1 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$:



Q: Express $\vec u$ as a lin. comb. of $\vec v_1, \vec v_2$.

Sol: by parallelogram rule, $\vec u = 2\vec v_1 + \vec v_2$.

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Ex: $\vec a_1 = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}$ $\vec a_2 = \begin{bmatrix} +3 \\ -5 \\ 2 \end{bmatrix}$ $\vec b = \begin{bmatrix} -1 \\ -1 \\ -4 \end{bmatrix}$

Q: can $\vec b$ be generated as a lin. comb. of $\vec a_1, \vec a_2$, i.e., do weights x_1, x_2 exist, s.t. $x_1 \vec a_1 + x_2 \vec a_2 = \vec b$? $(*)$

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Sol: $(*)$ means $x_1 \underbrace{\begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}}_{\vec a_1} + x_2 \underbrace{\begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}}_{\vec a_2} = \begin{bmatrix} -1 \\ -1 \\ -4 \end{bmatrix}$

lhs $= \begin{bmatrix} x_1 \\ -3x_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ -5x_2 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 \\ -3x_1 - 5x_2 \\ -x_1 + 2x_2 \end{bmatrix}$

So, $(*)$ is true iff $\begin{cases} x_1 + 3x_2 = -1 \\ -3x_1 - 5x_2 = -1 \\ -x_1 + 2x_2 = -4 \end{cases}$ - lin. sys. ! $(**)$

Aug. Mat.: $\begin{bmatrix} 1 & 3 & -1 \\ -3 & -5 & -1 \\ -1 & 2 & -4 \end{bmatrix} \underset{\text{row equiv}}{\sim} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 4 & -4 \\ 0 & 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \\ 0 & 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ Sol of $(**)$: $x_1 = 2$, $x_2 = -1$

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Thus: $2\begin{bmatrix}1\\-3\\-1\end{bmatrix} + (-1)\begin{bmatrix}3\\-5\\2\end{bmatrix} = \begin{bmatrix}-1\\-1\\-4\end{bmatrix}$

$2\,\vec{a}_1 + (-1)\,\vec{a}_2 = \vec{b}$

So, $\vec{b}$ is a lin. comb. of $\vec{a}_1, \vec{a}_2$.

Note: the augm. matrix of our system was $[\vec{a}_1\ \vec{a}_2\ \vec{b}]$ $\left[\underset{\vec{a}_1}{\begin{matrix}1\\-3\\-1\end{matrix}}\ \underset{\vec{a}_2}{\begin{matrix}3\\-5\\2\end{matrix}}\ \underset{\vec{b}}{\begin{matrix}-1\\-1\\-4\end{matrix}}\right] \overset{\text{notation}}{=} [\vec{a}_1\ \vec{a}_2\ \vec{b}]$

- Equation $x_1\vec{a}_1 + \dots + x_n\vec{a}_n = \vec{b}$ has same solution set as the lin. sys whose augm. mat is $\boxed{[\vec{a}_1 \dots \vec{a}_n\ \vec{b}]}$ (***)

  In particular, $\vec{b}$ can be generated as a lin. comb. of $\vec{a}_1 \dots \vec{a}_n$ iff there exists a sol. for the lin. sys. corresp. to matrix (***).

Def Let $\vec{v}_1 \dots \vec{v}_p$ be in $\mathbb{R}^n$. The set of all linear comb. of $\vec{v}_1 \dots \vec{v}_p$ is denoted

Span$\{\vec{v}_1, \dots, \vec{v}_p\}$ = "subset of $\mathbb{R}^n$ spanned (or "generated") by $\vec{v}_1 \dots \vec{v}_p$".

I.e. Span$\{\vec{v}_1, \dots, \vec{v}_p\}$ = set of all vectors that can be written in the form $\boxed{c_1\vec{v}_1 + \dots + c_p\vec{v}_p}$ with $c_1 \dots c_p$ scalars.

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vector $\vec{b}$ is in Span$\{\vec{v}_1 \dots \vec{v}_p\}$ iff vector equation $x_1\vec{v}_1 + \dots + x_p\vec{v}_p = \vec{b}$ has a solution

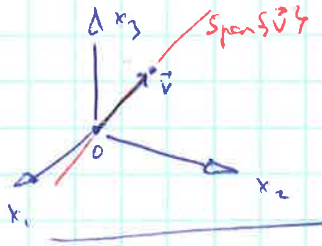
$\iff$ lin. sys. with aug. mat. $[\vec{v}_1 \dots \vec{v}_p\ \vec{b}]$ has a solution.

- Span$\{\vec{v}_1 \dots \vec{v}_p\}$ contains every scalar multiple of $\vec{v}_1$ and contains $\underbrace{\vec{0}}_{\text{in particular}}$.

Geom. description of Span$\{\vec{v}\}$ and Span$\{\vec{u}, \vec{v}\}$

let $\vec{v} \in \mathbb{R}^3$ ~~a~~ a nonzero vector

Span$\{\vec{v}\}$ = {scalar multiples of $\vec{v}$}
= points in the line in $\mathbb{R}^3$ through $\vec{v}$ and $\vec{0}$.



For $\vec{u}, \vec{v} \in \mathbb{R}^3$ nonzero vectors, s.t. $\vec{v}$ not a multiple of $\vec{u}$.
Then Span$\{\vec{u}, \vec{v}\}$ = plane in $\mathbb{R}^3$ that contains $\vec{u}, \vec{v}, \vec{0}$

Ex: Q: $\vec{a}_1 = \begin{bmatrix}1\\-2\\3\end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix}5\\-13\\-3\end{bmatrix}$, $\vec{b} = \begin{bmatrix}-3\\8\\1\end{bmatrix}$. Is $\vec{b}$ in the plane Span$\{\vec{a}_1, \vec{a}_2\}$?

Sol: does $x_1\vec{a}_1 + x_2\vec{a}_2$ have a solution?

$$\begin{bmatrix}1 & 5 & -3\\-2 & -13 & 8\\3 & -3 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 5 & -3\\0 & -3 & 2\\0 & -18 & 10\end{bmatrix} \sim \begin{bmatrix}1 & 5 & -3\\0 & -3 & 2\\\boxed{0 \quad 0 \quad -2}\end{bmatrix}$$

last eq.: 0 = -2 => no solutions
=> $\vec{b}$ is not in the plane!

1.4. The matrix equation $A\vec{x} = \vec{b}$

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def for an $m \times n$ matrix $A = [\underset{\nwarrow}{\vec{a}_1} \underset{\uparrow}{\dots} \underset{\nearrow}{\vec{a}_n}]$ and $\vec{x}$ a vector in $\mathbb{R}^n$,
columns

the matrix-vector product is:

$$A\vec{x} = [\vec{a}_1 \cdots \vec{a}_n]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} := \boxed{x_1\vec{a}_1 + \dots + x_n\vec{a}_n}$$

- Lin.comb. of columns of A with weights given by entries of $\vec{x}$.

Note: $A\vec{x}$ is defined only if $\boxed{\#\text{ columns in } A = \#\text{ entries in } \vec{x}}$

Ex:

$$\begin{bmatrix} 1 & -2 & 3 \\ -4 & 5 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -4 \end{bmatrix} + (-1)\begin{bmatrix} -2 \\ 5 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -8 \end{bmatrix} + \begin{bmatrix} 2 \\ -5 \end{bmatrix} + \begin{bmatrix} 9 \\ 0 \end{bmatrix} = \begin{bmatrix} 13 \\ -13 \end{bmatrix}$$

$\uparrow \vec{a}_1 \quad \uparrow \vec{a}_2 \quad \uparrow \vec{a}_3$

Q: for $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^m$, write lin.comb. $\vec{u} = 2\vec{v}_1 - 5\vec{v}_2 + 3\vec{v}_3$ as a matrix-vector product

A:

$$\vec{u} = \underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}}_{m\times 3 \text{ matrix}}\begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

Lin.sys.

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 5 \\ -4x_1 + 5x_2 &= 7 \end{aligned} \iff \overset{\text{vector eq.}}{x_1\begin{bmatrix} 1 \\ -4 \end{bmatrix} + x_2\begin{bmatrix} -2 \\ 5 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}} \iff \overset{\text{~~matrix~~ eq.}}{\boxed{\begin{bmatrix} 1 & -2 & 3 \\ -4 & 5 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}}}$$

matrix of coeffs of the lin.sys.

"matrix equation" of form $A\vec{x} = \vec{b}$

If $A = [\vec{a}_1 \cdots \vec{a}_n]$ is $m \times n$ matrix, $\vec{b} \in \mathbb{R}^m$, then matrix eq. $A\vec{x} = \vec{b}$ has same solution set as the vector equation $x_1\vec{a}_1 + \dots + x_n\vec{a}_n = \vec{b}$ which has in turn same sol. set as lin.sys. w/ augm. matrix $[\vec{a}_1 \cdots \vec{a}_n \; \vec{b}]$

Existence of solutions

Eq. $\boxed{A\vec{x} = \vec{b}}$ has a solution iff $\vec{b}$ is a lin.comb. of columns of A.
$(\iff \vec{b} \in \text{Span}\{\vec{a}_1 \cdots \vec{a}_n\})$

Ex: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.   Q: Is eq. $A\vec{x} = \vec{b}$ consistent for all $\vec{b}$?

Sol: Aug. Mat:

$$\begin{bmatrix} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & -6 & -12 & b_3 - 7b_1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & 0 & 0 & \underbrace{b_3 - 7b_1 - 2(b_2 - 4b_1)}_{b_1 - 2b_2 + b_3} \end{array}\right]$$

WANT THIS TO VANISH f. consistency

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⇒ $A\vec{x}=\vec{b}$ consistent iff $\underbrace{b_1-2b_2+b_3=0}$

equation of a plane through the origin in $\mathbb{R}^3$
$= \text{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$

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- Note $A\vec{x}=\vec{b}$ is not consistent for every $\vec{b}$, because REF of A has a row of zeros!

  If A had a pivot in each row, REF of $[A\ \vec{b}]$ would be of form $\begin{bmatrix} \bullet & * & * & * \\ 0 & \bullet & * & * \\ 0 & 0 & \bullet & * \end{bmatrix}$

  – consistent for every $\vec{b}$.

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AHM)

- Let A be an $m\times n$ matrix. The following are equivalent:
  - (a) For each $\vec{b}\in\mathbb{R}^m$, eq. $A\vec{x}=\vec{b}$ has a solution
  - (b) Each $\vec{b}\in\mathbb{R}^m$ is a lin. comb. of columns of A
  - (c) Columns of A span $\mathbb{R}^m$ ← entire
  - (d) A has a pivot in every row.

Note: ↑ coeff. matrix, not the aug. mat.

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Row-vector rule for computing $A\vec{x}$.

if $A\vec{x}$ product is defined, then i^{th} entry in $A\vec{x}$ is the sum of products of corresponding entries from row i of A and from vector $\vec{x}$.

Ex:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} x_1+2x_2+3x_3 \\ 4x_1+5x_2+6x_3 \\ 7x_1+8x_2+9x_3 \end{bmatrix}$$

"dot product"

$$\begin{bmatrix} 1 & 2 & 3 \\ & & \\ & & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} & & \\ 4 & 5 & 6 \\ & & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Ex:

$$\begin{bmatrix} 1 & -2 & 3 \\ -4 & 5 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1\cdot 2+(-2)(-1)+3\cdot 3 \\ (-4)\cdot 2+5\cdot(-1)+0\cdot 3 \end{bmatrix} = \begin{bmatrix} 13 \\ -13 \end{bmatrix}$$

- Properties of matrix-vector product $A\vec{x}$

  ~~ex~~ for A $m\times n$ mat. $\vec{u},\vec{v}\in\mathbb{R}^n$ and c a scalar:

  (a) $A(\vec{u}+\vec{v}) = A\vec{u}+B\vec{v}$

  (b) $A(c\vec{u}) = c\cdot(A\vec{u})$