$\Rightarrow$ $A\vec{x} = \vec{b}$ consistent iff $\underbrace{b_1 - 2b_2 + b_3 = 0}$

equation of a plane through the origin in $\mathbb{R}^3$

$= \text{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$

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- Note $A\vec{x} = \vec{b}$ is not consistent for every $\vec{b}$, because REF of A has a row of zeros!

If A had a pivot in each row, REF of $[A\,\vec{b}]$ would be of form $\begin{bmatrix} \bullet & * & * & * \\ 0 & \bullet & * & * \\ 0 & 0 & \bullet & * \end{bmatrix}$

- consistent for every $\vec{b}$.

(THM)

- Let A be an $m \times n$ matrix. The following are equivalent:
  - (a) For each $\vec{b} \in \mathbb{R}^m$, eq. $A\vec{x} = \vec{b}$ has a solution
  - (b) Each $\vec{b} \in \mathbb{R}^m$ is a lin. comb. of columns of A
  - (c) Columns of A span entire $\mathbb{R}^m$
  - (d) A has a pivot in every row.

Note: coeff. matrix, not the aug. mat.

Row-vector rule for computing $A\vec{x}$.

if product $A\vec{x}$ is defined, then i^{th} entry in $A\vec{x}$ is the sum of products of corresponding entries from row i of A and from vector $\vec{x}$.

Ex: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 4x_1 + 5x_2 + 6x_3 \\ 7x_1 + 8x_2 + 9x_3 \end{bmatrix}$

"dot product" $\begin{bmatrix} 1 & 2 & 3 \\ & & \\ & & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$\begin{bmatrix} \\ 4 & 5 & 6 \\ \\ \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Ex: $\begin{bmatrix} 1 & -2 & 3 \\ -4 & 5 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1\cdot 2 + (-2)(-1) + 3\cdot 3 \\ (-4)\cdot 2 + 5\cdot(-1) + 0\cdot 3 \end{bmatrix} = \begin{bmatrix} 13 \\ -13 \end{bmatrix}$

- Properties of matrix-vector product $A\vec{x}$

for A $m \times n$ mat. $\vec{u}, \vec{v} \in \mathbb{R}^n$ and c a scalar:

(a) $A(\vec{u} + \vec{v}) = A\vec{u} + B\vec{v}$

(b) $A(c\vec{u}) = c\cdot(A\vec{u})$

1.5. Solution sets of linear systems

$\underset{m\times n \text{ matrix}}{A}\vec{x} = \vec{0}$ - homogeneous system. $\vec{x} = \vec{0} \in \mathbb{R}^n$ always a solution - "trivial solution"

Q: are there non-trivial solutions?

Answer: Homogeneous equation $A\vec{x} = \vec{0}$ has a non-trivial sol. iff the eq. has at least one free variable

Ex 1:

$$2x_1 - x_2 + 3x_3 = 0$$
$$2x_1 + 2x_2 + 3x_3 = 0$$
$$-6x_1 \quad - 9x_3 = 0$$

Q: is there a nontriv. solution? describe the sol. set.

Solution: Aug. Mat: $\underset{\text{coeff. mat}}{[A\ \vec{0}]} = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 2 & 2 & 3 & 0 \\ -6 & 0 & -9 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -3 & 0 & 0 \end{bmatrix} \sim \begin{array}{c} \begin{bmatrix} 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} x_1 & x_2 & x_3 & \end{matrix} \end{array}$ REF

x_3 ↑ free variable
⇒ There are nontriv. solutions!

$\sim \begin{bmatrix} 1 & 0 & 3/2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ RREF

$$x_1 + \tfrac{3}{2}x_3 = 0$$
$$x_2 = 0$$
$$0 = 0$$

⇒ solve for basic vars

$x_1 = -\frac{3}{2}x_3$
$x_2 = 0$
x_3 free

⇒ sol. in vector form $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3/2\, x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} -3/2 \\ 0 \\ 1 \end{bmatrix}}_{=:\vec{v}} = x_3\vec{v}$

⇒ Solution set is the line $\boxed{\text{Span}\left\{\vec{v} = \begin{bmatrix} -3/2 \\ 0 \\ 1 \end{bmatrix}\right\}}$ in $\mathbb{R}^3$

Note: a nontriv. solution can have some (not all!) zero entries

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Ex 2: $3x_1 + x_2 - 6x_3 = 0$ - "system" of 1 eq. on 3 vars

Q: describe the solution set

Sol: x_1 basic x_2, x_3 free solve for x_1: $x_1 = -\frac{1}{3}x_2 + 2x_3$, x_2 free, x_3 free

implicit description of the plane

general sol: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}x_2 + 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} -1/3 \\ 1 \\ 0 \end{bmatrix}}_{\vec{u}} + x_3 \underbrace{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}}$

explicit "parametric vector form" of the plane
$\vec{x} = s\vec{u} + t\vec{v}$, $s, t \in \mathbb{R}$ parameters

Thus, sol. set is $\text{Span}\{\vec{u}, \vec{v}\}$ - a plane in $\mathbb{R}^3$

- Sol. set to any homog. eq. $\boxed{A\vec{x} = \vec{0}}$ has form $\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$ for suitable vectors $\vec{v}_1 \dots \vec{v}_p \in \mathbb{R}^n$

  Special case: sol. set $= \text{Span}\{\vec{0}\} = \{\vec{0}\}$ - case when there are no non-triv. solutions

Solutions of non-homogeneous systems

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Ex: describe all solutions of $A\vec{x} = \vec{b}$ with

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 2 & 2 & 3 \\ -6 & 0 & -9 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$$

matrix of coeff from Ex^1

Sol: $[A\ \vec{b}] = \begin{bmatrix} 2 & -1 & 3 & 3 \\ 2 & 2 & 3 & 0 \\ -6 & 0 & -9 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{3}{2} & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad x_1 = 1 - \frac{3}{2}x_3$, $x_2 = -1$, x_3 free

$$\Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 - \frac{3}{2}x_3 \\ -1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{3}{2}x_3 \\ 0 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{\vec{p}} + x_3 \underbrace{\begin{bmatrix} -3/2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}} = \vec{p} + x_3\vec{v}$$

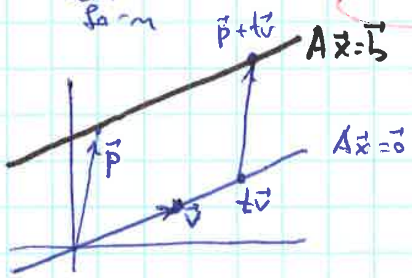
Sol. set in param. vector form: $\boxed{\vec{x} = \vec{p} + t\vec{v}}$, $t \in \mathbb{R}$ $\quad (t = x_3)$

recall: solutions set for $A\vec{x} = 0$ was $\vec{x} = t\vec{v}$, $t \in \mathbb{R}$

add $\vec{p}$ to a generic sol. of $A\vec{x} = 0$

same $\vec{v}$!

"addition = translation"



Sol. set for homog. eq $A\vec{x} = 0$ - a line through the origin

—— " —— nonhomog eq $A\vec{x} = \vec{b}$ - a parallel line passing through $\vec{p}$ (a fixed solution)

THM Suppose eq. $A\vec{x} = \vec{b}$ (*) is consistent for some given $\vec{b}$, and let $\vec{p}$ be a solution. Then, the sol. set of $A\vec{x} = \vec{b}$ is the set of vectors of form $\vec{w} = \vec{p} + \vec{v}_h$, where $\vec{v}_h$ is any sol. of the homogeneous eq. $A\vec{x} = \vec{0}$.

NB! If (*) is inconsistent, set of sol. is empty

THM applies to consistent eqs.

WRITING A SOL. SET in parametric form [THE ALGORITHM] $\quad$ 5,3

1. Aug. Mat. $\sim$ RREF
2. solve for basic vars in terms of free vars (if any)
3. write a general sol. $\vec{x}$ as a vector w/ entries depending on free vars
4. decompose $\vec{x}$ into a lin. comb. of vectors (with numeric entries) using free vars as parameters