Solutions of non-homogeneous systems

Ex: describe all solutions of $A\vec{x} = \vec{b}$ with

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 2 & 2 & 3 \\ -6 & 0 & -9 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$$

$\uparrow$ matrix of coeff from Ex 1

Sol: $[A \ \vec{b}] = \begin{bmatrix} 2 & -1 & 3 & 3 \\ 2 & 2 & 3 & 0 \\ -6 & 0 & -9 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{3}{2} & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad x_1 = 1 - \frac{3}{2}x_3$, $x_2 = -1$, x_3 free

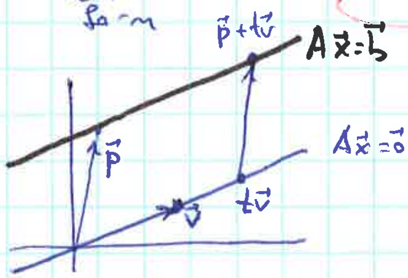
$$\Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 - \frac{3}{2}x_3 \\ -1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{3}{2}x_3 \\ 0 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{\vec{p}} + x_3 \underbrace{\begin{bmatrix} -3/2 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}} = \vec{p} + x_3\vec{v}$$

Sol. set in param. vector form $\boxed{\vec{x} = \vec{p} + t\vec{v}}$, $t \in \mathbb{R}$ $\quad (t = x_3)$

recall: solution set for $A\vec{x} = 0$ was $\vec{x} = t\vec{v}$, $t \in \mathbb{R}$

add $\vec{p}$ to a generic sol. of $A\vec{x} = 0$ $\qquad$ same $\vec{v}$!

"addition = translation"



Sol. set for homog eq $A\vec{x} = 0$ - a line through the origin

— " — nonhomog eq $A\vec{x} = \vec{b}$ - a parallel line passing through $\vec{p}$ (a fixed solution)

THM Suppose eq. $A\vec{x} = \vec{b}$ (*) is consistent for some given $\vec{b}$, and let $\vec{p}$ be a solution. Then, the sol. set of $A\vec{x} = \vec{b}$ is the set of vectors of form $\vec{w} = \vec{p} + \vec{v}_h$, where $\vec{v}_h$ is any sol. of the homogeneous eq. $A\vec{x} = \vec{0}$.

NB! If (*) is inconsistent, set of sol. is empty
THM applies to consistent eqs.

WRITING A SOL. SET in parametric form [THE ALGORITHM] $\qquad$ 5,3

1. Aug. Mat. $\sim$ RREF
2. solve for basic vars in terms of free vars (if any)
3. write a general sol. $\vec{x}$ as a vector w/ entries depending on free vars
4. decompose $\vec{x}$ into a lin. comb. of vectors (with numeric entries) using free vars as parameters

1.7. Linear dependence

def A set of vectors $\{\vec{v}_1, \dots, \vec{v}_p\}$ in $\mathbb{R}^n$ is linearly independent if the vector eq. $\boxed{x_1\vec{v}_1 + \dots + x_p\vec{v}_p = \vec{0}}$ has only the triv. solution.

Set $\{\vec{v}_1, \dots, \vec{v}_p\}$ is linearly dependent if there exist weights $c_1, \dots, c_p$ (not all zero), s.t.

$c_1\vec{v}_1 + \dots + c_p\vec{v}_p = \vec{0}$ — "linear dependence relation among $\vec{v}_1, \dots, \vec{v}_p$"

Ex: $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} -1 \\ 6 \\ 0 \end{bmatrix}$

Q: (1) Is the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ linearly dependent?

(2) If yes, find a lin. dep. relation.

Sol:

Vec. eq. $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$ (*)

Aug. mat.: $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 0 & 6 & 0 \\ 2 & 3 & 0 & 0 \end{array}\right] \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -4 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

REF: columns x_1, x_2, x_3; x_3 ← free var. RREF

(1) x_3 free $\Rightarrow$ (*) has (infinitely many) nontriv. solutions $\Rightarrow$ set-lin. dependent

(2) from RREF: $x_1 = -3x_3$
$x_2 = 2x_3$
x_3 free

e.g. for $x_3 = -4$, we have a sol. $(12, -8, -4)$

$\Rightarrow$ lin. dep. rel. $12\vec{v}_1 - 8\vec{v}_2 - 4\vec{v}_3 = \vec{0}$

– one of inf. many lin. dep. relations

Lin. independence of matrix columns

for $A = [\vec{a}_1 \dots \vec{a}_n]$ a matrix, eq. $A\vec{x} = \vec{0} \iff x_1\vec{a}_1 + \dots + x_n\vec{a}_n = \vec{0}$

So: • each lin. dep. rel. among columns of A corresponds to a nontriv. sol. of $A\vec{x} = \vec{0}$

• columns of A are lin. indep. iff eq. $A\vec{x} = \vec{0}$ has only the triv. sol.

Ex: $A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$ are the columns lin. indep.?

Sol: $A\vec{x} = \vec{0}$ Aug. Mat. $\begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{bmatrix}$

$\Rightarrow x_1, x_2, x_3$ – basic vars

$\Rightarrow$ no nontriv. sol.

$\Rightarrow$ columns of A are lin. indep.

Sets of one or two vectors

- $\{\vec{v}\}$ - set of 1 vector - is lin. indep. iff $\vec{v} \neq \vec{0}$

Ex: (a) $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -8 \\ -4 \end{bmatrix}$ (b) $\vec{v}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -8 \\ -4 \end{bmatrix}$ Q: Are these two sets lin. indep.?

Sol: (a) notice: $\vec{v}_2 = -4\vec{v}_1$ a scalar multiple $\Rightarrow 4\vec{v}_1 + \vec{v}_2 = \vec{0}$ lin. dep. rel.

(b) $\vec{v}_1$ and $\vec{v}_2$ are <u>not multiples</u> of one another. Assume $c\vec{v}_1 + d\vec{v}_2 = \vec{0}$ $(*)$ lin. dep. rel.

if $c \neq 0$, then $(*) \Rightarrow \vec{v}_1 = (-\frac{d}{c})\vec{v}_2$ - impossible. Thus $c = 0$

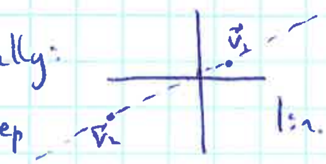
Similarly, $d = 0$. Hence in $(*)$ $c = d = 0$. ▨

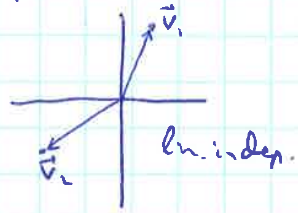
$\Rightarrow \{\vec{v}_1, \vec{v}_2\}$ lin. indep. set

- A set of two vectors $\{\vec{v}_1, \vec{v}_2\}$ is lin. dep. iff at least one of the vectors is a multiple of another.

Geometrically:

$\{\vec{v}_1, \vec{v}_2\}$ lin. dep iff they are on the same line through the origin.





Sets of ≥ 2 vectors

<u>Thm</u> ("characterization of lin. dep. sets") A set $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$, of ≥ 2 vectors, is lin. dep. iff at least one of the vectors in S is a lin. comb. of others.

In fact, if S is lin. dep. and $\vec{v}_1 \neq \vec{0}$, then some $\vec{v}_j$ (with $j > 1$) is a lin. comb. of preceding vectors $\vec{v}_1, \ldots, \vec{v}_{j-1}$

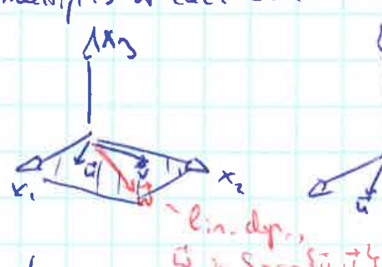
<u>WARNING</u> Thm does not say that <u>every</u> vector in S is a lin. comb of the others.

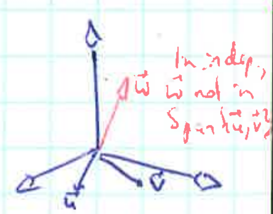
Ex: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ - lin. dep.

Ex: $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ Span $\vec{u}, \vec{v}$ <u>lin. indep.</u> (not multiples of each other)

Span$\{\vec{u}, \vec{v}\}$ = $x_1 x_2$ plane

- $\vec{w} \in$ Span$\{\vec{u}, \vec{v}\}$ iff $\{\vec{u}, \vec{v}, \vec{w}\}$ is a lin. dep. set (by THM)





<u>THM</u> If a set contains more vectors than there are entries in each vector, then the set is lin. dep. I.e. any set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ in $\mathbb{R}^n$ is lin. dep. if $p > n$.

Proof: $A = [\vec{v}_1 \ldots \vec{v}_p]$ - n equations on p variables, #eqs $<$ #vars $\Rightarrow$ there are free variables $\Rightarrow$ $A\vec{x} = \vec{0}$ admits a nontriv. sol. $\Rightarrow$ columns of A are lin. dep. ▨

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- If a set $S = \{\vec{v}_1, \dots, \vec{v}_p\}$ in $\mathbb{R}^n$ contains the <u>zero vector</u>, then S is ln. dep.

  (E.g. $\vec{v}_1 = \vec{0}$. Then $1\cdot\vec{v}_1 + 0\cdot\vec{v}_2 + \dots + 0\cdot\vec{v}_p = 0$ ln. dep. rel.)

Ex: (a) $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}$ – ln. dep. $(\underset{3}{p} > \underset{2}{n})$

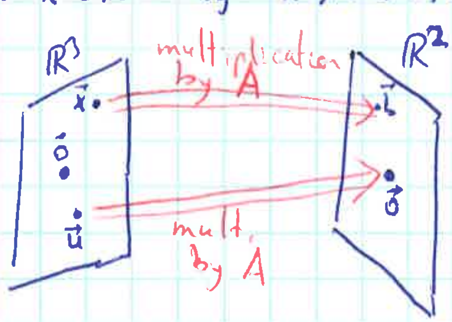
(b) $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}$ ln. dep.

(c) $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}3\\6\\9\\1\end{bmatrix}$ neither vector is a multiple of the other $\rightarrow$ ln. indep.

1.8. <u>Linear transformations.</u>

Ex: $A = \begin{bmatrix}1 & 2 & 4\\0 & 1 & 2\end{bmatrix}$ A "acts" on vectors $\vec{x} \in \mathbb{R}^3$ by transforming $\vec{x}$ into $A\vec{x} \in \mathbb{R}^2$

$A\underbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix}7\\3\end{bmatrix}}_{\vec{b}}$, $A\underbrace{\begin{bmatrix}0\\2\\-1\end{bmatrix}}_{\vec{u}} = \underbrace{\begin{bmatrix}0\\0\end{bmatrix}}_{\vec{0}}$



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A <u>transformation</u> (or <u>function</u>, or <u>mapping</u>) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule which assigns to each vector $\vec{x}$ in $\mathbb{R}^n$ a vector $T(\vec{x})$ in $\mathbb{R}^m$.

$\mathbb{R}^n$ – "domain" of T, $\mathbb{R}^m$ – "codomain" of T. Notation: $T: \underbrace{\mathbb{R}^n}_{\text{domain}} \rightarrow \underbrace{\mathbb{R}^m}_{\text{codomain}}$

for $\vec{x} \in \mathbb{R}^n$, $\underset{\in \mathbb{R}^m}{T(\vec{x})}$ – "image of $\vec{x}$" (under $\underbrace{T}_{\text{the action of}}$).

Range of T = set of all images $T(\vec{x})$.

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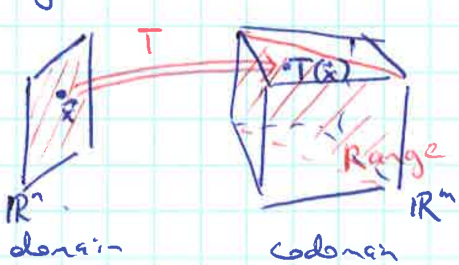

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given A an $m \times n$ matrix, we have a <u>matrix transformation</u> $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $T(\vec{x}) = A\vec{x}$.

<u>Notation</u> for such a transf.: $\vec{x} \longmapsto A\vec{x}$

n: # columns, m: # rows