1/29/2018
3

- If a set $S=\{\vec{v}_1,\dots,\vec{v}_p\}$ in $\mathbb{R}^n$ contains the <u>zero vector</u>, then S is lin. dep.

(E.g. $\vec{v}_1=\vec{0}$. Then $1\cdot\vec{v}_1+0\cdot\vec{v}_2+\dots+0\cdot\vec{v}_p=0$ lin. dep. rel.)

Ex: (a) $\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}2\\3\end{bmatrix},\begin{bmatrix}3\\4\end{bmatrix}$ - lin. dep. $(\underset{3}{p}>\underset{2}{n})$

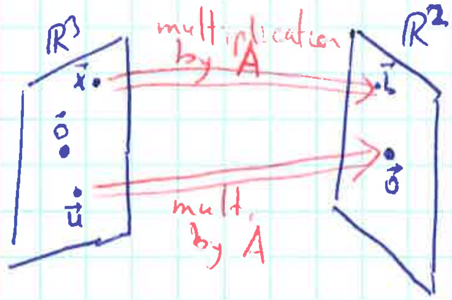
(b) $\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix},\begin{bmatrix}0\\0\\0\end{bmatrix}$ lin. dep.

(c) $\begin{bmatrix}1\\2\\3\\4\end{bmatrix},\begin{bmatrix}3\\6\\9\\1\end{bmatrix}$ neither vector is a multiple of the other $\to$ lin. indep.

1.8. Linear transformations.

Ex: $A=\begin{bmatrix}1&2&4\\0&1&2\end{bmatrix}$ A "acts" on vectors $\vec{x}\in\mathbb{R}^3$ by transforming $\vec{x}$ into $A\vec{x}\in\mathbb{R}^2$

$\underset{\vec{x}}{A\begin{bmatrix}1\\1\\1\end{bmatrix}}=\underset{\vec{b}}{\begin{bmatrix}7\\3\end{bmatrix}}$, $\underset{\vec{u}}{A\begin{bmatrix}0\\2\\-1\end{bmatrix}}=\underset{\vec{0}}{\begin{bmatrix}0\\0\end{bmatrix}}$



5

1/31/2018
0

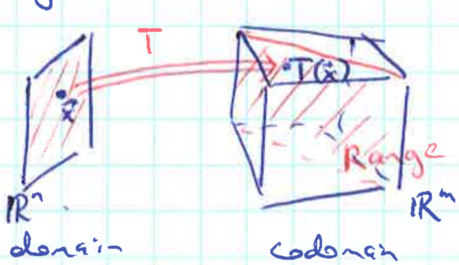
A <u>transformation</u> (or <u>function</u>, or <u>mapping</u>) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule which assigns to each vector $\vec{x}$ in $\mathbb{R}^n$ a vector $T(\vec{x})$ in $\mathbb{R}^m$

$\mathbb{R}^n$ - "domain" of T, $\mathbb{R}^m$ - "codomain" of T. Notation: $T:\underbrace{\mathbb{R}^n}_{\text{domain}}\to\underbrace{\mathbb{R}^m}_{\text{codomain}}$

for $\vec{x}\in\mathbb{R}^n$, $\underset{\in\mathbb{R}^m}{T(\vec{x})}$ - "image of $\vec{x}$" (under $\underbrace{T}_{\text{the action of}}$).

Range of T = set of all images $T(\vec{x})$.

3



---

given A an $m\times n$ matrix, we have a <u>matrix transformation</u> $T:\mathbb{R}^n\to\mathbb{R}^m$ with $T(\vec{x})=A\vec{x}$.

Notation for such a transf.: $\vec{x}\longmapsto A\vec{x}$

#columns (n)   #rows (m)

1/31/2018
1

Ex $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, define $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\vec{x}) = A\vec{x}$

(a) find image $T(\vec{u})$ under T, for $\vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

__Sol__: $A\vec{u} = \begin{bmatrix} 1 \cdot 2 + (-3)(-1) \\ 3 \cdot 2 + 5(-1) \\ (-1)2 + 7(-1) \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$

(b) Solve $T(\vec{x}) = \underbrace{\begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}}_{\vec{b}}$ for $\vec{x}$.

__Sol__: Aug. Mat of $A\vec{x} = \vec{b}$: $\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 = -3/2 \\ x_2 = 1/2 \end{matrix}$

$\Rightarrow \vec{x} = \begin{bmatrix} -3/2 \\ 1/2 \end{bmatrix}$ ← its image is $\vec{b}$!

uniqueness question

(c) Is there more than one $\vec{x}$ s.t. $T(\vec{x}) = \vec{b}$?

Sol: $A\vec{x} = \vec{b}$ has unique sol. (no free vars) $\Rightarrow$ NO, there is only one $\vec{x}$ whose image is $\vec{b}$

existence question

(d) Is $\vec{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ in the range of T?

Sol: $A\vec{x} = \vec{c}$; does sol exist? Aug. Mat. $\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ \boxed{0 \quad 0 \quad -35} \end{bmatrix}$

system inconsistent $\Rightarrow$ sol. does not exist

$\Rightarrow$ $\boxed{\text{NO}}$

__def__ A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is *linear* if

(i) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v} \in \mathbb{R}^n$

(ii) $T(c\vec{u}) = c\,T(\vec{u})$ for all $c \in \mathbb{R}$, $\vec{u} \in \mathbb{R}^n$.

__Main Ex__: every matrix transf. ~~$A\vec{x} \to$~~ $T : \vec{x} \mapsto A\vec{x}$ is a lin. transf.:

indeed, $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$, $A(c\vec{u}) = c(A\vec{u})$

- Properties of a lin. transf.: $T(\vec{0}) = \vec{0}$

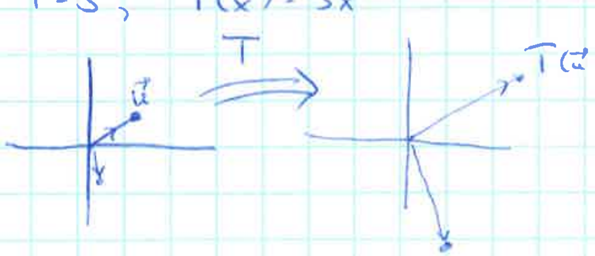
$$T(c\vec{u} + d\vec{v}) = c\,T(\vec{u}) + d\,T(\vec{v})$$

more generally $T(c_1\vec{v}_1 + \dots + c_p\vec{v}_p) = c_1 T(\vec{v}_1) + \dots + c_p T(\vec{v}_p)$ ← in engineering: "superposition principle"

Ex: for r a scalar, define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\vec{x}) = r\vec{x}$. for $0 \leq r < 1$, T is called "contraction"

for $r > 1$, T is called "dilation"

E.g. $r = 3$, $T(\vec{x}) = 3\vec{x}$



T is linear!

Check: $T(c\vec{u} + d\vec{v}) = 3 \cdot (c\vec{u} + d\vec{v})$

$= c(3\vec{u}) + d(3\vec{v})$

$= c\,T(\vec{u}) + d\,T(\vec{v})$ ☑

1.9. Matrix of a lin. transf.

1/31/2018

Ex: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ its columns: $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose $T: \mathbb{R}^2 \to \mathbb{R}^3$ lin. transf. s.t.

$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$. Find $T(\vec{x})$ for arbitrary $\vec{x} \in \mathbb{R}^2$.

Sol: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1 \vec{e}_1 + x_2 \vec{e}_2 \Rightarrow T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2) \overset{\text{linearity}}{=} x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2)$

$$= x_1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}}_{T(\vec{e}_1)} + x_2 \underbrace{\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}}_{T(\vec{e}_2)} = \underbrace{\boxed{\begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 3 & 5 \end{bmatrix}}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\vec{x}$$

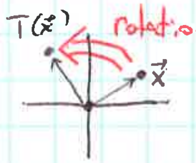
THM Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a lin. transf. Then there exists a unique mat A s.t. $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

In fact, $\underbrace{A = [T(\vec{e}_1) \dots T(\vec{e}_n)]}_{\text{stand. matrix of lin. transf. } T}$ where $\vec{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j^{\text{th}}$ place is the j-th column of identity matrix I_n in $\mathbb{R}^n$.

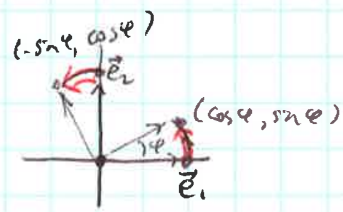
Ex: for T the dilation with $r=3$, $T(\vec{e}_1) = 3\vec{e}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $T(\vec{e}_2) = 3\vec{e}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ - stand. matrix of T.

Ex: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $T: \mathbb{R}^2 \to \mathbb{R}^2$
$\vec{x} \mapsto A\vec{x}$

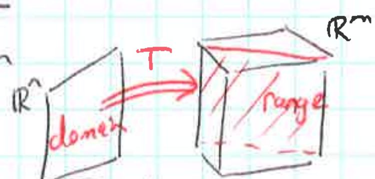


$A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$ - stand. mat. of rotation by angle φ

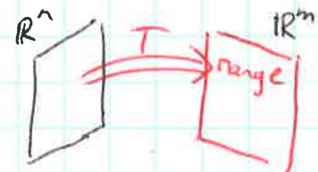


def A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is onto $\mathbb{R}^m$ if each $\vec{b} \in \mathbb{R}^m$ is the image of at least one $\vec{x} \in \mathbb{R}^n$.

- T is onto iff range = codomain



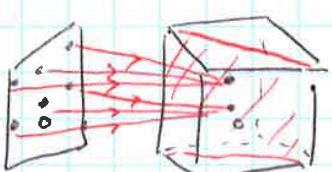
T not onto

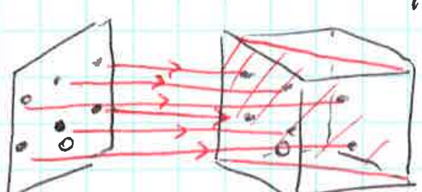


T onto $\mathbb{R}^m$

def $T: \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each $\vec{b} \in \mathbb{R}^m$ is the image of at most one $\vec{x} \in \mathbb{R}^n$

- T is 1-1 iff $T(\vec{x}) = \vec{b}$ for each $\vec{b}$ has a unique sol., or none at all.


T not 1-1


T 1-1

Ex: T - lin. mapping with stand mat $A = \begin{bmatrix} 1 & 2 & -3 & 2 \\ 0 & 4 & 1 & -1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ Q: (a) is $T: \mathbb{R}^4 \to \mathbb{R}^3$ onto?
(b) is T one-to-one?

Sol: (a) ~~piv~~ A has a pivot in every row $\Rightarrow A\vec{x} = \vec{b}$ consistent $\forall \vec{b} \Rightarrow T$ is onto.
(b) $A\vec{x} = \vec{b}$ has a free variable $\Rightarrow$ no uniqueness $\Rightarrow$ T not one-to-one!

$\dfrac{1/31/2018}{3}$

• T is one-to-one iff eq. $T(\vec{x}) = \vec{0}$ has only the triv. sol.

<u>Thm</u> Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a lin. transf., A the stand. mat.

(a) T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ the columns of A span $\mathbb{R}^m$

(b) T is one-to-one iff the columns of A are lin. indep.