02/09/2018
2

Claim: if H has a basis of p vectors, then each basis in H has exactly p vectors.

def The dimension of a nonzero subspace H, dim H, is the number of vectors in any basis for H. Also, $\dim\{\vec{0}\} = 0$ (convention)

Ex: • $\dim \mathbb{R}^n = n$, every basis in $\mathbb{R}^n$ consists of n vectors.

• dim(plane in $\mathbb{R}^3$ through 0) = 2 • dim (line through 0 in $\mathbb{R}^3$) = 1.

Ex: Nul A : basis vectors correspond to free variables of $A\vec{x}=\vec{0}$.

So, dim Nul A = # non-pivotal columns = # free variables.

Def The rank of a mat. A, rank A, is dim Col A.

Thus, rank A = # pivot columns

Ex: $A = \begin{bmatrix} 1 & 2 & 1 & 7 & 1 \\ -2 & -4 & -1 & -10 & -2 \\ 3 & 6 & 0 & 9 & 1 \\ 1 & 2 & -2 & -5 & 7 \end{bmatrix}$ Q: What is rank A?

Sol: $A \sim \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ 3 pivot columns => rank = 3

Note: dim Nul A = # non-pivot col. = 2

THM (The rank theorem)

If a matrix A has n columns, then rank A + dim Nul A = n

Thm (basis theorem)

Let H be a p-dimensional subspace of $\mathbb{R}^n$. Any lin. indep. set of p vectors in H is automatically a basis for H. Also, any set of p vectors in H spanning H, is a basis for H.

4.1 5,3

02/12/2018
0

Invertible mat. THM (continued)

A n×n mat. Following are equivalent to A being invertible:

(m) Columns of A form a basis for $\mathbb{R}^n$

(n) Col A = $\mathbb{R}^n$

(o) dim Col A = n

(p) rank A = n

(q) Nul A = $\{\vec{0}\}$

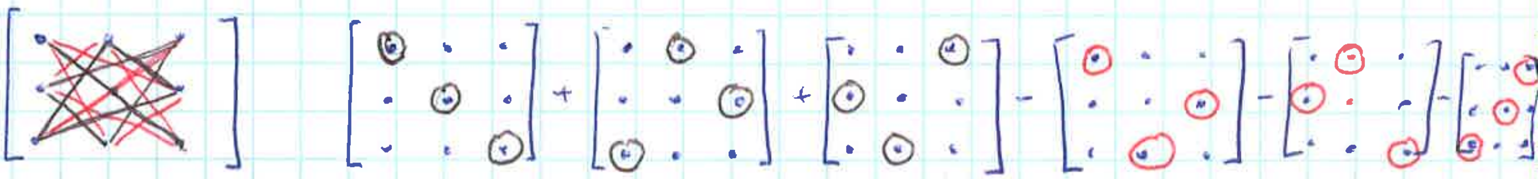
(r) dim Nul A = 0.

3.1 Determinants

Recall: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is invertible iff $\underbrace{a_{11}a_{22} - a_{12}a_{21}}_{\det A} \neq 0$ - "determinant"

for a 3×3 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \det A := a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$



A invertible iff $\det A \neq 0$.

- $\det A = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$

$$= a_{11}\det\underbrace{\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}}_{A_{11}} - a_{12}\det\underbrace{\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}}_{A_{12}} + a_{13}\det\underbrace{\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}}_{A_{13}} = a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13}$$

A_{11} - A with 1st row and 1st column deleted

for A $n\times n$ square matrix, A_{ij} - submatrix formed by deleting row i and column j from A

Ex: $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$ $\qquad A_{23} = \begin{bmatrix} 1 & 2 & 4 \\ 9 & 10 & 12 \\ 13 & 14 & 16 \end{bmatrix}$

- for a 1×1 matrix, $\det[a_{11}] = a_{11}$
- recursive definition of the determinant:

Def for $A = [a_{ij}]$ an $n\times n$ matrix, $n \geq 2$, the determinant is:

(*) $\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + \dots + (-1)^{1+n} a_{1n}\det A_{1n} = \sum_{j=1}^{n} (-1)^{1+j} a_{1j}\det A_{1j}$

Ex: $A = \begin{bmatrix} 2 & 3 & 0 \\ 5 & 1 & -7 \\ 0 & 0 & -2 \end{bmatrix}$ Q: compute $\det A$.

Sol: $\det A = 2\underbrace{\begin{vmatrix} 1 & -7 \\ 0 & -2 \end{vmatrix}}_{=\det(\cdot\cdot)\text{ - notation}} - 3\begin{vmatrix} 5 & -7 \\ 0 & -2 \end{vmatrix} + 0\cdot\begin{vmatrix} 5 & 1 \\ 0 & 0 \end{vmatrix} = 2\cdot(-2) - 3(-10) + 0\cdot 0 = 26$

For $A=[a_{ij}]$, the $\boxed{(i,j)\text{-cofactor}}$ of A is $\boxed{C_{ij} := (-1)^{i+j} A_{ij}}$

We have $(*)$ $\det A = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$ - cofactor expansion across the first row of A

THM $\det A$ can be computed by a cofactor expansion across any row,

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in} \qquad (\text{row } i)$$

or down any column:

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \dots + a_{nj}C_{nj} \qquad (\text{column } j)$$

Ex: $A = \begin{bmatrix} 2 & 5 & 0 \\ 3 & 1 & 0 \\ 0 & -7 & -2 \end{bmatrix}$ Q: Compute $\det A$ (by cofactor expansion)

Sol: ~~det~~ cofactor expansion down 3rd column:

$$\det A = (-2)\begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} + 0\cdot\begin{vmatrix} 3 & 1 \\ 0 & -7 \end{vmatrix} + 0\cdot\begin{vmatrix} 2 & 5 \\ 0 & -7 \end{vmatrix} =$$

$$\left(= a_{31}\underbrace{C_{31}}_{A_{31}} + a_{32}\underbrace{C_{32}}_{-A_{32}} + a_{33}\underbrace{C_{33}}_{A_{33}}\right)$$

$$= (-2)\cdot(-13) = 26.$$

· It is useful to do cofactor expansion across row/col. containing many zeros!

Ex: $A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$

$$\underset{\uparrow \text{1st col.}}{\det A =} 3\begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} + 0\cdot(\ldots) + 0\cdot(\ldots) + 0\cdot(\ldots) + 0\cdot(\ldots)$$

$$\underset{\uparrow \text{1st col.}}{=} 3\cdot 2\begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} \underset{\uparrow \text{3rd row}}{=} 3\cdot 2\cdot \underset{(-1)^{3+2}}{(-1)}(-2)\underbrace{\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix}}_{-1} = -12$$

THM: If A is a triangular matrix, then $\det A$ is the product of diagonal entries of A.

Ex: $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} = 1\cdot\begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = 1\cdot 4\cdot |6| = 1\cdot 4\cdot 6$

Practice problem: Compute $\begin{vmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{vmatrix}$