3.2. Properties of determinants

how det changes under row operations?

THM

Let A be a square matrix

(a) if $A \underset{\text{row replacement}}{\sim} B$, then $\det B = \det A$

(b) if $A \underset{\text{row interchange}}{\sim} B$, then $\det B = -\det A$

(c) if $A \underset{\substack{r \mapsto k\cdot r \\ \text{row}}}{\sim} B$, then $\det B = k \cdot \det A$.

Ex: compute $\det A$, $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$

Sol: $\det A = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} \underset{\text{row replacements}}{=} \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix} \underset{r_2 \rightleftarrows r_3}{=} - \underset{\text{triangular!}}{\begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix}} = -1 \cdot 3 \cdot (-5)$

- Suppose A was reduced to REF U using only row replacements & interchanges (always possible!)

Then: $\det A = (-1)^{\underset{\#\text{ interchanges}}{r}} \det U \underset{\substack{\text{since } U \\ \text{REF} \Rightarrow \text{triangular}}}{=} (-1)^r \cdot u_{11} \cdots u_{nn}$

$U = \begin{bmatrix} \bullet & * & * & * \\ 0 & \bullet & * & * \\ 0 & 0 & \bullet & * \\ 0 & 0 & 0 & \bullet \end{bmatrix}$ A invertible $\Rightarrow$ $\det A = (-1)^r \cdot \begin{pmatrix}\text{product} \\ \text{of pivots}\end{pmatrix}$

$U = \begin{bmatrix} \bullet & * & * & * \\ 0 & \bullet & * & * \\ 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 \end{bmatrix}$ A not invertible $\Rightarrow$ $\det A = 0$

THM A square matrix A is invertible iff $\boxed{\det A \neq 0}$.

- $\det A = 0 \iff$ columns of A are lin. dep. $\iff$ rows of A are lin. ~~indep~~ dep.

Ex: $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 3 & 4 \end{vmatrix} = 0$ because $r_3 = r_1 + r_2$.

Ex: (combining row reduction and cofactor expansion)

$$\begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{vmatrix} \underset{r_4 \mapsto r_4 + r_2}{=} \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & -5 & -7 & -3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix} \underset{\substack{\text{cofactor} \\ \text{exp. down} \\ \text{col. 1}}}{=} -2 \begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix} \underset{r_2 \mapsto r_2 - 3r_1}{=} -2 \begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{vmatrix}$$

$$\underset{r_2 \rightleftarrows r_3}{=} 2 \underbrace{\begin{vmatrix} 1 & 2 & -1 \\ 0 & -3 & 1 \\ 0 & 0 & 5 \end{vmatrix}}_{\text{triangular}} = 2 \cdot 1 \cdot (-3) \cdot 5 = -30$$

THM: $\boxed{\det A^T = \det A}$

THM (multiplicative property): $\det(AB) = (\det A)(\det B)$

Ex: $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ $\quad B = \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix}$ $\quad AB = \begin{bmatrix} 2 & 5 \\ 4 & 13 \end{bmatrix}$

$\det A = 3$ $\quad \det B = 2$ $\quad \det AB = 2 \cdot 13 - 5 \cdot 4 = 6 = 2 \cdot 3$ ✓

WARNING: Generally, $\det(A+B) \neq \det A + \det B$.

(in Ex above, $\det(A+B) = \begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix} = 2 \neq \cancel{\det}\ 3 + 2$)

- One can perform column operations on A, similar to row operations; $\det A$ changes in the same way as for row operations.

Linearity property of the determinant function.

Suppose j-th col. of A can vary: $A = [\vec{a}_1 \cdots \vec{a}_{j-1}\ \vec{x}\ \vec{a}_{j+1} \cdots \vec{a}_n]$

Set $T: \mathbb{R}^n \longrightarrow \mathbb{R}$

$\vec{x} \longmapsto T(\vec{x}) = \det[\vec{a}_1 \cdots \vec{a}_{j-1}\ \vec{x}\ \vec{a}_{j+1} \cdots \vec{a}_n]$

Then: $T(c\vec{x}) = c\,T(\vec{x})$

$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$. I.e., T is a linear mapping.

Practice problem: Compute $\begin{vmatrix} 1 & -3 & 1 & -2 \\ 2 & -5 & -1 & -2 \\ 0 & -4 & 5 & 1 \\ -3 & 10 & -6 & 8 \end{vmatrix}$