2.1 Matrix operations

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \leftarrow \text{row } i$$

$\vec{a}_1 \quad \vec{a}_j \quad \vec{a}_n$

column j

(i,j)-entry of A

Zero matrix 0 (of size $m\times n$) - all entries zeroes.

- for A, B of same size $m\times n$, can form a **sum** $A+B$, $(A+B)_{ij} = a_{ij}+b_{ij}$

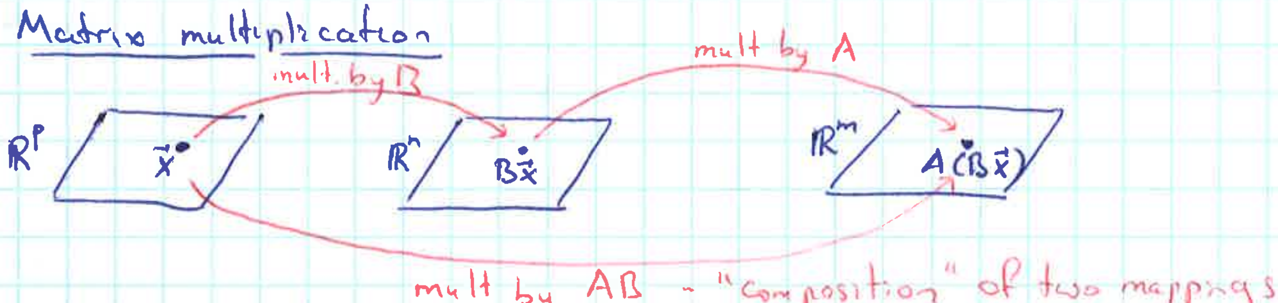
Ex: $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$ Then $A+B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & -1 \end{bmatrix}$

- can also form scalar multiples cA, $(cA)_{ij} = c\cdot a_{ij}$ (c ← scalar)

Ex: $2B = \begin{bmatrix} 0 & -2 & 6 \\ 2 & 4 & 0 \end{bmatrix}$, $A+2B = \begin{bmatrix} 1 & 1 & 6 \\ 4 & 8 & -1 \end{bmatrix}$

properties - as for vector operations: $c(A+B) = cA + cB$ etc.

Matrix multiplication



WANT a matrix AB s.t. $A(B\vec{x}) = (AB)\vec{x}$ for any $\vec{x}$

$$A(B\vec{x}) = A(x_1\vec{b}_1 + \dots + x_p\vec{b}_p) = x_1A\vec{b}_1 + \dots + x_pA\vec{b}_p = \underbrace{[A\vec{b}_1 \cdots A\vec{b}_p]}_{AB}\vec{x}$$

def If A is an $m\times n$ matrix, B $n\times p$ matrix then the product AB is an $m\times p$ matrix, $AB = A[\vec{b}_1 \cdots \vec{b}_p] = \boxed{[A\vec{b}_1 \cdots A\vec{b}_p]}$.

- multiplication of matrices corresponds to composition of lin. transf.

Ex: $A = \begin{bmatrix} 1 & 3 \\ -2 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 5 \end{bmatrix}$ ($b_1 \; b_2 \; b_3$) ~~AS~~ Q: Compute AB

Sol: $A\vec{b}_1 = \begin{bmatrix} 1 & 3 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$, $Ab_2 = \begin{bmatrix} 10 \\ -5 \end{bmatrix}$, $Ab_3 = \begin{bmatrix} 15 \\ -5 \end{bmatrix}$ $\Rightarrow AB = \begin{bmatrix} 2 & 10 & 15 \\ -4 & -5 & -5 \end{bmatrix}$

- for AB to be defined, need $\#\text{cols}(A) = \#\text{rows}(B)$

 if AB defined, $\#\text{rows}(AB) = \#\text{rows}(A)$, $\#\text{cols}(AB) = \#\text{cols}(B)$

 $\boxed{(m\times n\text{-matrix})\cdot(n\times p\text{-matrix}) = (m\times p \text{ matrix})}$

- Row-column rule for AB: $(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$

Ex: $\begin{bmatrix} 1 & 3 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 5 \end{bmatrix} = \begin{bmatrix} * & * & \boxed{1\cdot 0+3\cdot 5} \\ * & * & * \end{bmatrix}$ entry (1,3)

1st row

3rd column

$\begin{bmatrix} 1 & 3 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 5 \end{bmatrix} = \begin{bmatrix} * & * & * \\ * & \boxed{(-2)1+(-1)3} & * \end{bmatrix}$ entry (2,2)

Properties: $A(BC) = (AB)C$, $(A+B)C = \cancel{AB}\ AC + BC$, $\boxed{I_m A = A = A I_n}$

- Generally, $AB \neq BA$ (when both defined)

Ex: $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$

$AB = \begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix}$ $BA = \begin{bmatrix} 10 & 2 \\ 29 & -2 \end{bmatrix}$, $AB \neq BA$!

WARNINGS
1. Generally, $AB \neq BA$
2. Cancellation laws don't hold: $AB = AC \not\Rightarrow B = C$
3. $AB = 0 \not\Rightarrow A = 0$ or $B = 0$

- If A an $n\times n$ matrix, $k \geq 1$, $A^k = \underbrace{A\cdot A\cdot\dots\cdot A}_{k}$ - k-th power of A.

- Transpose

 for A an $m\times n$ matrix, its transpose A^T is an $n\times m$ matrix whose columns are formed from respective rows of A.

Ex: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \Rightarrow B^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

Properties $(A^T)^T = A$, $(A+B)^T = A^T + B^T$, $(cA)^T = cA^T$, $\boxed{(AB)^T = B^T A^T}$

reverse order!

2.2 The inverse of a matrix

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A $n\times n$ mat. is <u>invertible</u> if there is an $n\times n$ mat. C s.t. $C A = \underbrace{I}_{I_n}$ and $AC = I$.

Then C is called <u>the inverse of A</u>.

It is <u>unique</u> (if exists), notation: A^{-1}. Thus $A^{-1}A = I$, $AA^{-1} = I$.

a non-invertible A is called <u>"singular"</u>

Ex: $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$ $AC = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $CA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Thus, $C = A^{-1}$.

<u>THM</u> Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $\boxed{ad-bc} = 0$, then A is non-invertible

"determinant", det A

Ex: det $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ $\det A = 2(-7) - 5(-3) = 1$, $A^{-1} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$ cf. prev. Ex.

Ex: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

- If A is an <u>invertible</u> $n\times n$ matrix, then for each $\vec{b} \in \mathbb{R}^n$, eq. $A\vec{x} = \vec{b}$ has the unique sol. $\boxed{\vec{x} = A^{-1}\vec{b}}$

- properties: • $(A^{-1})^{-1} = A$ • $\boxed{(AB)^{-1} = \underbrace{B^{-1}A^{-1}}_{\text{reverse order!}}}$ • $(A^T)^{-1} = (A^{-1})^T$

<u>Elementary matrices</u>

an <u>elem. matrix</u> is the result of a single elem. row operation on the identity matrix.

Ex: $E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ $E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$

$r_2 \mapsto r_2 - 3r_1$ $r_1 \rightleftarrows r_2$ $r_2 \mapsto 5r_2$

for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $E_1 A = \begin{bmatrix} a & b \\ c-3a & d-3b \end{bmatrix}$ $E_2 A = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$ $E_3 A = \begin{bmatrix} a & b \\ 5c & 5d \end{bmatrix}$

- If an elem. row op. is performed on $m\times n$ mat. A, the resulting matrix is EA, where E is the $m\times m$ elem. mat. created by doing same row op. on I_m

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$A \underset{\rho}{\sim} \cancel{E}A$, where $I \underset{\rho}{\sim} E$

- each E is invertible; the inverse is an elem. matrix of same type.

THM An $n \times n$ matrix is invertible iff $A \sim I_n$. In this case, any sequence of row op. that reduces A to I_n, also transforms I_n to A^{-1}!

Algorithm for finding A^{-1}

Row reduce the augmented matrix $[A \; I]$ ($n \times 2n$ - matrix) | If A is invertible, ~~then A~~ the RREF is $[I \; A^{-1}]$

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad [A \; I] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{bmatrix} \sim \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$

$\underbrace{\quad}_{I} \; \underbrace{\quad}_{A^{-1}}$