4.4 Coordinate systems

a basis $\mathcal{B}$ (with n vectors) for V imposes a "coord. system" on V, which makes V "act like $\mathbb{R}^n$".

THM (The unique representation thm)

Let $\mathcal{B}$ ~~be a~~ $=\{\vec{b}_1,\dots,\vec{b}_n\}$ be a basis for a v.sp. V. Then for each $\vec{x}$ in V, there exists a unique set of scalars $c_1,\dots,c_n$ s.t. $\boxed{\vec{x}=c_1\vec{b}_1+\dots+c_n\vec{b}_n}$ $(*)$

from spanning property of $\mathcal{B}$ — from LI property

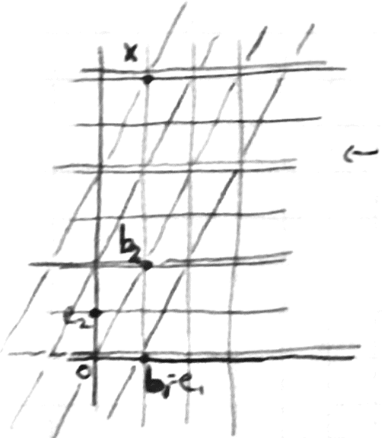
def Weights $c_1,\dots,c_n$ in $(*)$ are the coordinates of $\vec{x}$ rel. to $\mathcal{B}$ ($\mathcal{B}$-coordinates of $\vec{x}$)

$$[\vec{x}]_{\mathcal{B}}=\begin{bmatrix}c_1\\ \vdots\\ c_n\end{bmatrix}\in\mathbb{R}^n$$ - coord. vector of $\vec{x}$ (rel. to $\mathcal{B}$) / $\mathcal{B}$-coord. vector of $\vec{x}$

Mapping $$\begin{aligned} V &\longrightarrow \mathbb{R}^n\\ \vec{x} &\longmapsto [\vec{x}]_{\mathcal{B}}\end{aligned}$$ - coord. mapping defined by $\mathcal{B}$.

Ex: $\mathcal{B}=\left\{\underset{b_1}{\begin{bmatrix}1\\0\end{bmatrix}},\underset{b_2}{\begin{bmatrix}1\\2\end{bmatrix}}\right\}$ - basis in $\mathbb{R}^2$. Q: $\vec{x}\in\mathbb{R}^2$ with $[\vec{x}]_{\mathcal{B}}=\begin{bmatrix}-2\\3\end{bmatrix}$ find $\vec{x}$

Sol: $\vec{x}=-2\vec{b}_1+3\vec{b}_2=\begin{bmatrix}1\\6\end{bmatrix}$. Note: $\vec{x}=\begin{bmatrix}1\\6\end{bmatrix}=1\cdot\vec{e}_1+6\vec{e}_2$, thus coord. vect of $\vec{x}$ rel. to stand. basis $\mathcal{E}=\{\vec{e}_1,\vec{e}_2\}$ is $\boxed{[\vec{x}]_{\mathcal{E}}=\vec{x}}$



← stand. coordinate grid on $\mathbb{R}^2$ and $\mathcal{B}$-coord. grid

Coordinates in $\mathbb{R}^n$

Ex: $\mathcal{B}=\left\{\underset{b_1}{\begin{bmatrix}2\\1\end{bmatrix}},\underset{b_2}{\begin{bmatrix}-1\\1\end{bmatrix}}\right\}$ in $\mathbb{R}^2$ $\vec{x}=\begin{bmatrix}4\\5\end{bmatrix}$ Q: find $[\vec{x}]_{\mathcal{B}}$

Sol: $c_1\vec{b}_1+c_2\vec{b}_2=\vec{x} \iff \underbrace{\begin{bmatrix}2&-1\\1&1\end{bmatrix}}_{P_{\mathcal{B}}}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}4\\5\end{bmatrix}$ sol: $[\vec{x}]_{\mathcal{B}}=\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix}$

$P_{\mathcal{B}}$ - matrix changing $\mathcal{B}$-coords of $\vec{x}$ from to stand. coordinates

- For any basis $\mathcal{B}$ in $\mathbb{R}^n$, let $P_{\mathcal{B}}=[\vec{b}_1\ \dots\ \vec{b}_n]$

then $\vec{x}=c_1\vec{b}_1+\dots+c_n\vec{b}_n \iff \vec{x}=P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} \implies P_{\mathcal{B}}^{-1}\vec{x}=[\vec{x}]_{\mathcal{B}}$

change-of-coord. mat. from $\mathcal{B}$ to $\mathcal{E}$ — mat. of coord. mapping $\vec{x}\mapsto[\vec{x}]_{\mathcal{B}}$ one-to-one, onto

Coord. mapping

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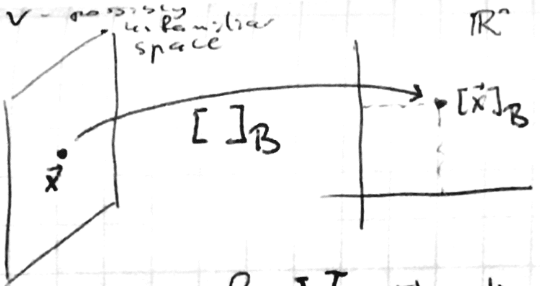

- "new name for $\vec{x}$"

THM: Let $\mathcal{B} = \{\vec{b}_1, \ldots \vec{b}_n\}$ be a basis for V. Then the coord. mapping $\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$ is a one-to-one linear transf. from V onto $\mathbb{R}^n$

- In particular, $[c_1\vec{u}_1 + \ldots + c_p\vec{u}_p]_{\mathcal{B}} = c_1[\vec{u}_1]_{\mathcal{B}} + \ldots + c_p[\vec{u}_p]_{\mathcal{B}}$ - preserves lin. comb.
- A lin. mapping $T: V \to W$ which is onto and 1-1, is called an isomorphism.
  - every vector space calculation in V is accurately reproduced in W and vice versa

  So, V and W are "same".
- A vect. sp. V with a basis $\mathcal{B}$ of n vectors is indistinguishable from $\mathbb{R}^n$.

Ex: $\mathcal{B} = \{1, t, t^2, t^3\}$ stand. basis in $\mathbb{P}_3$

p in $\mathbb{P}_3$ is $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ - lin. comb. of stand. basis vectors $\qquad [p]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$.

Coord mapping $\quad p \mapsto [p]_{\mathcal{B}}$

$\mathbb{P}_3 \to \mathbb{R}^4 \quad$ - $\mathbb{P}_3$ "~~looks/feel~~ acts" like $\mathbb{R}^4$

Ex: Check that $p_1 = 1+2t^2$, $p_2 = 4+t+5t^2$, $p_3 = 3+2t$ are LD in $\mathbb{P}_2$, using coord. vectors.

Sol: Aug Mat of $A\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 5 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{col of } A \text{ are lin. dep.}$$

$[p_1]_{\mathcal{B}}\ [p_2]_{\mathcal{B}}\ [p_3]_{\mathcal{B}}$

$$\sim \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} x_1 - 5x_3 = 0 \\ x_2 + 2x_3 = 0 \end{matrix} \Rightarrow \overbrace{[p_3]_{\mathcal{B}} = -5[p_1]_{\mathcal{B}} + 2[p_2]_{\mathcal{B}}}^{\text{lin. dependence of columns}}$$

$$\Rightarrow \underbrace{3+2t}_{p_3} = -5\underbrace{(1+2t^2)}_{p_1} + 2\underbrace{(4+t+5t^2)}_{p_2}$$ - relation for polynomials

Ex: $\vec{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$ $\qquad \mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ is a basis for $H = \operatorname{Span}\{\vec{v}_1, \vec{v}_2\}$

(a) is $\vec{x}$ in H? $\quad$ (b) if yes, find $[\vec{x}]_{\mathcal{B}}$.

Sol: $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{x}$ $\quad$ Aug. Mat. $\begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ Sol. exists, $c_1 = 2$, $c_2 = 3$ $\qquad$ (a) - YES $\quad$ (b) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

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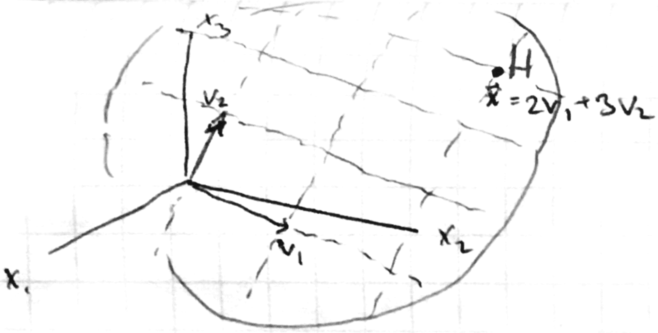

Practice question: ① $\mathcal{B} = \{p_1 = 1+2t^2,\ p_2 = t,\ p_3 = 2+3t^2\}$ in $\mathbb{P}_2$, $p = 1+t+t^2$

~~find $P_{\mathcal{B}}$~~, Q: find $[p]_{\mathcal{B}}$.

② $\mathcal{B} = \left\{ \underset{b_1}{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}, \underset{b_2}{\begin{bmatrix} 2 \\ 5 \end{bmatrix}} \right\}$ in $\mathbb{R}^2$

(a) find $P_{\mathcal{B}}$

(b) $P_{\mathcal{B}}^{-1} = ?$

(c) find $\mathcal{B}$-coord. vector of $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ using (b).