4.5. <u>The dimension of a vector space</u>

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Recall: for V - v.sp., $\mathcal{B}$ - basis with n vectors, coord. mapping $V \to \mathbb{R}^{n}$ - isomorphism

(n - dimension (intrinsic property) of V)

THM *

If a v.sp. V has a basis $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$, then any set of $p > n$ vectors in V must be **LD**.

(<u>Idea of proof</u>: $\{[V_1]_{\mathcal{B}}, \dots, [V_p]_{\mathcal{B}}\}$ - set of $p > n$ vectors in $\mathbb{R}^n$ $\Rightarrow$ LD in $\mathbb{R}^n$ $\Rightarrow$ $\{V_1 \dots V_p\}$ LD in V)

THM If a v.sp. V has a basis of n vectors, then every basis of V has exactly n vectors

(<u>Idea</u>: Let $\mathcal{B}_1$ - basis of n vectors, $\mathcal{B}_2^{p}$ - basis of p vectors, $p > n$.
By THM*, $\mathcal{B}_2^{p}$ - LD set $\Rightarrow$ not a basis - contradiction!)

<u>Recall</u>: If V is spanned by a finite S, then a subset of S is a basis of V

<u>def</u>. If V is spanned by a finite set, then V is <u>finite-dimensional</u>,
dimension $\dim V$ = number of vectors in a basis for V

- $\dim \{\vec{0}\} = 0$ (convention)
- if V is not spanned by a finite set, then V is <u>infinite-dimensional</u>

<u>Ex</u>: - Stand basis for $\mathbb{R}^n$ consists of n vectors, so $\dim \mathbb{R}^n = n$.

- For $\mathbb{P}_2$, $\{1, t, t^2\}$ - stand. basis, so $\dim \mathbb{P}_2 = 3$. Generally, $\dim \mathbb{P}_n = n+1$.
- $\mathbb{P}$ is infinite-dimensional

<u>Ex</u>: $H = \mathrm{Span}\left\{ \underset{v_1}{\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}} \right\}$ H - plane in $\mathbb{R}^3$ $\{\vec{v}_1, \vec{v}_2\}$ - basis for H hence $\dim H = 2$.

<u>Ex</u>: $H = \left\{ \begin{bmatrix} a + 3b + 6c \\ 2a + 5d \\ 4b + 8c - d \\ 9d \end{bmatrix} \middle| \ a, b, c, d \in \mathbb{R} \right\}$

Find <u>$\dim H$</u>.

<u>Sol</u>: $H = \mathrm{Span}\{\vec{v}_1, \vec{v}_2, \cancel{\vec{v}_3}, \vec{v}_4\}$

$$\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 4 \\ 0 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \\ 8 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ -1 \\ 9 \end{bmatrix}$$

Note: $V_3 = 2V_2$ $\xRightarrow{\text{Spanning set THM}}$ $\underbrace{\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}}_{\text{LI set}}$ spans H $\Rightarrow$ $\{\vec{v}_1, \vec{v}_2, \vec{v}_4\}$ basis for H

$\Rightarrow \dim H = 3$.

$\mathcal{E}x$ subspaces of $\mathbb{R}^3$; classified by dimension

- 0-dimensional $\{\vec{0}\}$
- 1-dim., $H = \text{Span}\{\vec{v}\}$ ← nonzero — lines through $\vec{0}$.
- 2-dim, $H = \text{Span}\underbrace{\{\vec{u}, \vec{v}\}}_{\text{LI set}}$ — planes through $\vec{0}$.
- 3-dim: $\mathbb{R}^3$ itself.

Subspaces of a fin. dim. space

THM** Let H be a subspace of a fin. dim. v.sp. V. Any LI set H can be expanded, if necessary, to a basis for H.
Also, H is fin. dim., and $\dim H \leq \dim V$.

Idea: ~~Let~~ $\{\vec{v}_1, \dots, \vec{v}_p\} = S$ - LI set in H. If S spans H, we are done: S - basis, $p \leq \dim V$ by THM*.
If S does not span H, take a vector $\vec{v}_{p+1}$ in $H \setminus \text{Span}\, V$ and adjoin to S, $S \to \{\vec{v}_1, \dots, \vec{v}_{p+1}\}$. Repeat until we span entire H.

THM (The Basis THM) Let V be a p-dimensional v.sp., $p \geq 1$. Then
- any LI set of p vectors in V is a basis for V. ← from THM**
- any spanning set for V of p vectors is a basis for V. ← from Spanning set THM

Recall: $\dim \text{Col}\, A = \#$ pivot columns in A

$\dim \text{Nul}\, A = \#$ free var. in $A\vec{x} = \vec{0}$ $= \#$ non-pivot columns in A.

$\mathcal{E}x$:

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3×5 matrix REF

$\text{Col}\, A \subset \mathbb{R}^3$
dim = 2

$\text{Nul}\, A \subset \mathbb{R}^5$
dim = 3

Practice problem: $H = \text{Span}\{\sin^2 t, \cos^2 t, 1\}$ in $C[0,1]$

(a) find a basis for H. $\dim H = ?$

(b) Is $f = \cos 2t$ in H? (c) If yes, find $[\cos 2t]_{\mathcal{B}}$ ← basis from (a)