2.8. Subspaces

def A <u>subspace</u> of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ s.t.
- (a) $\vec{0} \in H$
- (b) $\vec{u}+\vec{v} \in H$ for any pair $\vec{u},\vec{v} \in H$
- (c) $c\vec{u} \in H$ for any $\vec{u} \in H$, c scalar.

Ex: $\vec{v}_1,\vec{v}_2 \in \mathbb{R}^n$, then $H = \text{Span}\{\vec{v}_1,\vec{v}_2\}$ is a subspace of $\mathbb{R}^n$!

Check: (a) $0 = 0\cdot\vec{v}_1 + 0\cdot\vec{v}_2 \in H$

(b) $\vec{u} = s_1\vec{v}_1 + s_2\vec{v}_2$, $\vec{v} = t_1\vec{v}_1 + t_2\vec{v}_2$ $\Rightarrow$ $\vec{u}+\vec{v} = (s_1+t_1)\vec{v}_1 + (s_2+t_2)\vec{v}_2 \in H$ – ln. comb. of $\vec{v}_1,\vec{v}_2$

(c) $c\cdot\vec{u} = (cs_1)\vec{v}_1 + (cs_2)\vec{v}_2 \in H$.

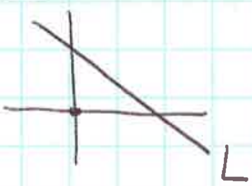
Thus: for $\vec{v}_1 \neq 0$, $\vec{v}_2 \neq c\cdot\vec{v}_1$, $\text{Span}\{\vec{v}_1,\vec{v}_2\}$ – a plane in $\mathbb{R}^n$ through 0 ← example of a subspace

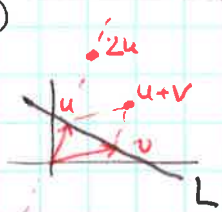
$\vec{v}_1 \neq 0$, $\vec{v}_2 = c\vec{v}_1$, $\text{Span}\{\vec{v}_1,\vec{v}_2\}$ – a line in $\mathbb{R}^n$ through 0

Ex a line L <u>not through the origin</u> is <u>not a subspace</u>.

(a, b and c all fail)

$0 \notin L$, $\vec{u}+\vec{v} \notin L$, $2\vec{u} \notin L$





Ex for $\vec{v}_1,\dots,\vec{v}_p \in \mathbb{R}^n$, $H = \text{Span}\{\vec{v}_1,\dots,\vec{v}_n\}$ is a subspace of $\mathbb{R}^n$ – the subspace "spanned" (or "generated") by $\vec{v}_1,\dots,\vec{v}_p$.

Ex: $\boxed{\mathbb{R}^n}$ itself is a subspace. Also, $\boxed{H=\{0\}}$ is a subspace ("zero subspace").

Column space and null space of a matrix

- A <u>column space</u> of a matrix A is the set of all linear comb. of the columns of A (Col A)

for $A = [\vec{a}_1 \cdots \vec{a}_n]$ $m\times n$ mat., $\boxed{\text{Col } A = \text{Span}\{\vec{a}_1,\dots,\vec{a}_n\}}$ – subspace of $\mathbb{R}^m$

- $\text{Col } A = \mathbb{R}^m$ iff columns of A span $\mathbb{R}^m$ $\Leftrightarrow$ pivot in each row of A.

Ex: $A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$ $\vec{b} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$ Q: Is $\vec{b}$ in Col A?

Sol: $A\vec{x} = \vec{b}$ consistent? Aug Mat $= \begin{bmatrix} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\Rightarrow$ eq. consistent $\Rightarrow$ YES.

Col A = set of all $\vec{b}$ s.t. $A\vec{x}=\vec{b}$ has a solution.

def The <u>null space</u> of A, Nul A, is the set of all sol. of homog. eq. $A\vec{x}=\vec{0}$.

- Nul A is a <u>subspace</u> of $\mathbb{R}^n$.

Nul A ~~defines a subspace~~ is defined <u>implicitly</u> (as solutions of an eq.), Col A is defined <u>explicitly</u>.

<u>Basis for a subspace</u>

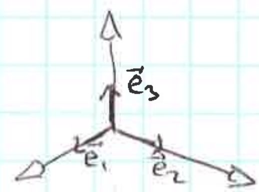
want to describe a subspace by the smallest possible set spanning it.

def A <u>basis</u> for a subspace H of $\mathbb{R}^n$ is a <u>ln. indep.</u> set in H which spans H.

Ex: columns of an invertible $n\times n$ matrix A form a basis for $H=\mathbb{R}^n$ (they are ln. indep & span $\mathbb{R}^n$ by Inv. Mat. THM)

E.g. for $A=I_n$, its columns $\vec{e}_1=\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \vec{e}_2=\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}, \dots, \vec{e}_n=\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}$

Set $\{\vec{e}_1,\dots,\vec{e}_n\}$ - <u>standard basis for $\mathbb{R}^n$</u>



Ex Find a <u>basis for Nul A</u>,

$$A=\begin{bmatrix}-3 & 6 & -1 & 1 & -7\\ 1 & -2 & 2 & 3 & -1\\ 2 & -4 & 5 & 8 & -4\end{bmatrix}$$

Sol. Write the sol. of $A\vec{x}=\vec{0}$ in parametric form:

$$\text{Aug Mat } [A\ \vec{0}] \sim \begin{array}{c}\begin{bmatrix}1 & -2 & 0 & -1 & 3 & 0\\ 0 & 0 & 1 & 2 & -2 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}\\ \underbrace{\quad x_2 \qquad\quad x_4 \quad x_5\quad}_{\text{free}}\end{array} \text{ RREF}$$

$$\begin{aligned} x_1 - 2x_2 \quad - x_4 + 3x_5 &= 0\\ x_3 + 2x_4 - 2x_5 &= 0\\ 0 &= 0\end{aligned}$$

Solution: $x_1 = 2x_2 + x_4 - 3x_5$
$x_3 = -2x_4 + 2x_5$
x_2, x_4, x_5 free

$$\Rightarrow \vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}2x_2+x_4-3x_5\\x_2\\-2x_4+2x_5\\x_4\\x_5\end{bmatrix}=x_2\underbrace{\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix}}_{\vec{u}}+x_4\underbrace{\begin{bmatrix}1\\0\\-2\\1\\0\end{bmatrix}}_{\vec{v}}+x_5\underbrace{\begin{bmatrix}-3\\0\\2\\0\\1\end{bmatrix}}_{\vec{w}}$$

$\vec{x} = x_2\vec{u} + x_4\vec{v} + x_5\vec{w}$

Thus: Nul A = Span$\{\vec{u},\vec{v},\vec{w}\}$. Moreover, $\vec{u},\vec{v},\vec{w}$ are ln. indep. ($x_2\vec{u}+x_4\vec{v}+x_5\vec{w}=0 \Rightarrow x_2,x_4,x_5=0$ by construction)

So, $\{\vec{u},\vec{v},\vec{w}\}$ - <u>basis</u> for Nul A.

Ex: RREF

$$B = \begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} b_1 & b_2 & b_3 & b_4 & b_5 \end{array} \end{array}$$

Q: find a basis for Col B.

Sol: note: $b_2 = 2b_1$, $b_4 = 3b_1 + 4b_3$

So, any lin. comb. of $b_1 .. b_5$ is in fact a lin. comb of b_1, b_3, b_5 (pivot columns)

$$\vec{v} = c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3 + c_4\vec{b}_4 + c_5\vec{b}_5 = c_1\vec{b}_1 + c_2 2\vec{b}_1 + c_3\vec{b}_3 + c_4(3\vec{b}_1 + 4\vec{b}_3) + c_5\vec{b}_5$$
$$= \cancel{(c_1 + 3c_4)\vec{b}_1}\ (c_1 + 2c_2 + 3c_4)\vec{b}_1 + (c_3 + 4c_4)\vec{b}_3 + c_5\vec{b}_5$$
$$\in \operatorname{Span}\{\vec{b}_1, \vec{b}_3, \vec{b}_5\}$$

Also, $\vec{b}_1, \vec{b}_3, \vec{b}_5$ are columns of I_4 and thus are lin. indep

$\Rightarrow \{\vec{b}_1, \vec{b}_3, \vec{b}_5\}$ - basis for Col B.

Ex:

$$A = \begin{array}{c} \begin{bmatrix} 1 & 2 & 1 & 7 & 1 \\ -2 & -4 & -1 & -10 & -2 \\ 3 & 6 & 0 & 9 & 1 \\ 1 & 2 & -2 & -5 & 7 \end{bmatrix} \\ \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \end{array} \sim B$$

pivot columns

Q: Find a basis of Col A.

Sol:

A lin. dependence rel. among col. of A is a sol. of $A\vec{x} = \vec{0}$. and $A\vec{x} = 0$, has same solutions as $B\vec{x} = 0$!

So: $b_2 = 2b_1$, $b_4 = 3b_1 + 4b_3$ $\Rightarrow$ $a_2 = 2a_1$, $a_4 = 3a_1 + 4a_3$

b_1, b_3, b_5 lin. indep $\Rightarrow$ a_1, a_3, a_5 lin. indep.

Thus: $\{a_1, a_3, a_5\}$ is a basis for Col A.

THM Pivot columns of A form a basis for Col A.

Warning: we need pivot columns of A itself, not of a REF of A.