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6.1 Inner product, length and orthogonality

> want to generalize geometric notions of length, distance, perpendicularity from $\mathbb{R}^2, \mathbb{R}^3$ to $\mathbb{R}^n$

Def For $\vec{u}, \vec{v} \in \mathbb{R}^n$, the inner product ("dot product") is $\underbrace{\vec{u}^T}_{1\times n \text{ mat}} \underbrace{\vec{v}}_{n\times 1 \text{ mat}} =: \vec{u}\cdot\vec{v}$ - a number

If $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, $\vec{u}\cdot\vec{v} = [u_1 \cdots u_n]\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \boxed{u_1v_1 + u_2v_2 + \dots + u_nv_n}$

Ex: $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}$

$\vec{u}\cdot\vec{v} = \vec{u}^T\vec{v} = [1\ 2\ 3]\begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} = 1\cdot3 + 2\cdot5 + 3(-1) = 10$

$\vec{v}\cdot\vec{u} = \vec{v}^T\vec{u} = [3\ 5\ -1]\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 3\cdot1 + 5\cdot2 + (-1)\cdot3 = 10$

THM: for $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, $c \in \mathbb{R}$,

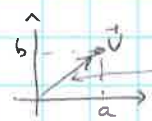
(a) $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
(b) $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}$
(c) $(c\vec{u})\cdot\vec{v} = c(\vec{u}\cdot\vec{v})$

$\Rightarrow (c_1\vec{u}_1 + \dots + c_p\vec{u}_p)\cdot\vec{v} = c_1(\vec{u}_1\cdot\vec{v}) + \dots + c_p(\vec{u}_p\cdot\vec{v})$

(d) $\vec{u}\cdot\vec{u} \geq 0$ and $\vec{u}\cdot\vec{u} = 0$ iff $\vec{u} = \vec{0}$.

Def Length ("norm") of $\vec{v} \in \mathbb{R}^n$ is $\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{v_1^2 + \dots + v_n^2} \geq 0$, $\|\vec{v}\|^2 = \vec{v}\cdot\vec{v}$

Ex: $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$



$\|\vec{v}\| = \sqrt{a^2+b^2}$ = length of the line segment (Pythagorean Thm)

- $\|c\vec{v}\| = |c|\,\|\vec{v}\|$ for $c \in \mathbb{R}$.
- a vector of length 1 - "unit vector". For $\vec{v} \neq \vec{0}$, $\vec{v} \longrightarrow \vec{u} = \frac{1}{\|\vec{v}\|}\vec{v}$ unit vector in the direction of $\vec{v}$

normalizing $\vec{v}$

Ex $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ Q: find $\vec{u}$ a unit vector in the same direction as $\vec{v}$

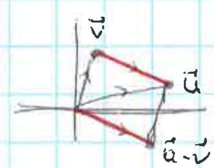
Sol: $\|\vec{v}\|^2 = \vec{v}\cdot\vec{v} = 1^2 + (-2)^2 + 2^2 = 9$, $\|\vec{v}\| = 3$, $\vec{u} = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$

Check: $\|\vec{u}\|^2 = \left(\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \frac{1+4+4}{9} = 1$

Def for $\vec{u}, \vec{v} \in \mathbb{R}^n$, the distance between $\vec{u}$ and $\vec{v}$ is

$\boxed{\text{dist}(\vec{u}, \vec{v}) := \|\vec{u} - \vec{v}\|}$

Ex: $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ $\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \left\|\begin{bmatrix} -2 \\ -3 \end{bmatrix}\right\| = \sqrt{(-2)^2 + (-3)^2} = \sqrt{13}$

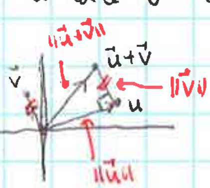


Orthogonal vectors

def Vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are orthogonal (perpendicular) (to each other) if $\boxed{\vec{u}\cdot\vec{v} = 0}$

- $\vec{u}$ and $\vec{v}$ are orthogonal iff $\underbrace{\|\vec{u}+\vec{v}\|^2}_{(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})} = \|\vec{u}\|^2 + \|\vec{v}\|^2$ ← Pythagorean thm

$\vec{u}\cdot\vec{u} + 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v} = \vec{u}\cdot\vec{u} + \vec{v}\cdot\vec{v}$



- Note: $\vec{0} \perp \vec{u}$ for any $\vec{u}$.

Orthogonal Complements

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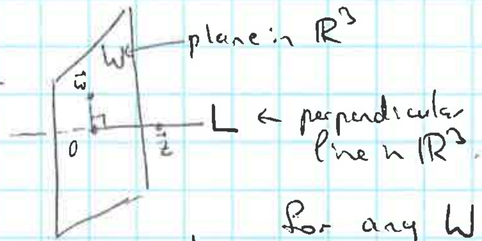
- If $\vec{z} \in \mathbb{R}^n$ is orthogonal to every vector in $W \subset \mathbb{R}^n$ (a subspace), then $\vec{z}$ is <u>orthogonal to W</u>

Set of all vectors in $\mathbb{R}^n$ orthogonal to W – "<u>orthogonal complement of W</u>", $W^\perp$ – notation

Ex



$$W^\perp = L, \quad L^\perp = W$$

- $\vec{x}$ is in $W^\perp$ iff $\vec{x}$ is orthog. to every vector in a set which spans W (for any W)
- $W^\perp$ is a <u>subspace</u> of $\mathbb{R}^n$.

Ex for A $m \times n$ mat., $\quad$ Nul A and Row A $\subset \mathbb{R}^n$ – orthogonal complements of each other

Nul A^T and Col A $\subset \mathbb{R}^m$ – orthogonal complements of each other.

- for $W \subset \mathbb{R}^n$, $\boxed{\dim W + \dim W^\perp = n}$

6.2 Orthogonal sets

A set of vectors $\{\vec{u}_1, \dots, \vec{u}_p\}$ in $\mathbb{R}^n$ is an <u>orthogonal set</u>, if $\vec{u}_i \cdot \vec{u}_j = 0$ for each pair $i \neq j$.

Ex*: $\vec{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$ $\quad$ Q: show that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthog. set

Sol: $\vec{u}_1 \cdot \vec{u}_2 = 3(-1) + 1 \cdot 2 + 1 \cdot 1 = 0$ ✓, $\vec{u}_1 \cdot \vec{u}_3 = 3(-\frac{1}{2}) + 1 \cdot (-2) + 1 \cdot \frac{7}{2} = 0$ ✓ $\quad \vec{u}_2 \cdot \vec{u}_3 = (-1)(-\frac{1}{2}) + 2(-2) + \frac{7}{2} = 0$ ✓

THM If $S = \{\vec{u}_1, \dots, \vec{u}_p\}$ an orthog. set of nonzero vectors in $\mathbb{R}^n$, then S is <u>lin. indep.</u> and hence a basis for Span S

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- an <u>orthogonal basis</u> for a subspace $W \subset \mathbb{R}^n$ is a basis for W which is also an orthogonal set.

THM* Let $\{\vec{u}_1, \dots, \vec{u}_p\}$ be an orthog. basis for $W \subset \mathbb{R}^n$. For each $\vec{y} \in W$, weights in $\vec{y} = c_1 \vec{u}_1 + \dots + c_p \vec{u}_p$ are given by $\boxed{c_j = \dfrac{\vec{y} \cdot \vec{u}_j}{\vec{u}_j \cdot \vec{u}_j}}$, $j = 1 \dots p$.

(Indeed: $\vec{y} \cdot \vec{u}_1 = c_1 \vec{u}_1 \cdot \vec{u}_1 + \cancel{c_2 \vec{u}_2 \cdot \vec{u}_1} + \dots + \cancel{c_p \vec{u}_p \cdot \vec{u}_1} \Rightarrow c_1 = \dfrac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}$ and similarly for other c_j)

Ex: orthog. set $S = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ from Ex* is a basis for $\mathbb{R}^3$, $\vec{y} = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$ $\quad$ Q: Express $\vec{y}$ as a lin. comb. of vectors in S

Sol: $\vec{y} = \underbrace{\dfrac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}}_{\frac{11}{11}} \vec{u}_1 + \underbrace{\dfrac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2}}_{\frac{-12}{6}} \vec{u}_2 + \underbrace{\dfrac{\vec{y} \cdot \vec{u}_3}{\vec{u}_3 \cdot \vec{u}_3}}_{\frac{-33}{33/2}} \vec{u}_3 = \vec{u}_1 - 2\vec{u}_2 - 2\vec{u}_3$ ← did not need to solve the lin. system to compute the weights!