02/28/2018
3

To solve two mat. eq. simultaneously, augment coeff. mat. with $\vec{b}_1$ and $\vec{b}_2$:

$$[\vec{c}_1\ \vec{c}_2 \mid \vec{b}_1\ \vec{b}_2] = \left[\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right]$$

03/02/2018
1

Thus: $[\vec{b}_1]_{\mathcal{C}} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, $[\vec{b}_2]_{\mathcal{C}} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ and $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$ 5

Observe: $[\vec{c}_1\ \vec{c}_2 \mid \vec{b}_1\ \vec{b}_2] \sim [I \mid \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}]$ ← works analogously for any two bases in $\mathbb{R}^n$ 3

Another description of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$:

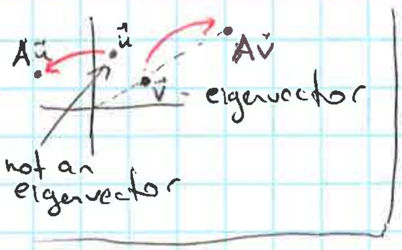
- $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{C}\leftarrow\mathcal{E}}{P}\ \underset{\mathcal{E}\leftarrow\mathcal{B}}{P} = (P_{\mathcal{C}})^{-1}P_{\mathcal{B}}$

or:
$$\vec{x} = P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}$$
$$\vec{x} = P_{\mathcal{C}}[\vec{x}]_{\mathcal{C}} \Rightarrow [\vec{x}]_{\mathcal{C}} = P_{\mathcal{C}}^{-1}\vec{x}$$
$$\Rightarrow [\vec{x}]_{\mathcal{C}} = \underbrace{P_{\mathcal{C}}^{-1}P_{\mathcal{B}}}_{\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}}[\vec{x}]_{\mathcal{B}}$$

5.1 Eigenvectors and eigenvalues

Def An eigenvector of an $n\times n$ matrix A is a nonzero vector $\vec{x}$ s.t. $\boxed{A\vec{x} = \lambda\vec{x}}$
German for "own", "proper"
some scalar

A scalar λ is called an eigenvalue of A if $A\vec{x} = \lambda\vec{x}$ has a nontriv. sol. $\vec{x}$.
Such $\vec{x}$ is called an eigenvector corresponding to λ.



$\mathcal{E}x^{(*)}$ $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ $\vec{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ **Q**: are $\vec{u}, \vec{v}$ eigenvectors?

Sol: $A\vec{u} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = (-4)\vec{u} \Rightarrow \vec{u}$ - eigenvector with $\lambda = -4$ the eigenvalue

$A\vec{v} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda\vec{v} \Rightarrow \vec{v}$ not an eigenvector!

$\mathcal{E}x$: Show that $\lambda = 7$ is an eigen~~vector~~value for A', find corresp. eigenvectors.

Sol: $\lambda = 7$ is an e-value iff $A\vec{x} = 7\vec{x}$ has a nontriv. sol. $\Leftrightarrow A\vec{x} - 7\vec{x} = \vec{0} \Leftrightarrow \boxed{(A-7I)\vec{x} = \vec{0}}$

$A - 7I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$ - col. LD $\Rightarrow$ there are nontriv. sol. to homog. eq.
$\Rightarrow \lambda = 7$ is an eigenvalue!

Aug. Mat $\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, general sol: $\vec{x} = x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ - each such vector with $x_2 \neq 0$ is an eigenvector for $\lambda = 7$.

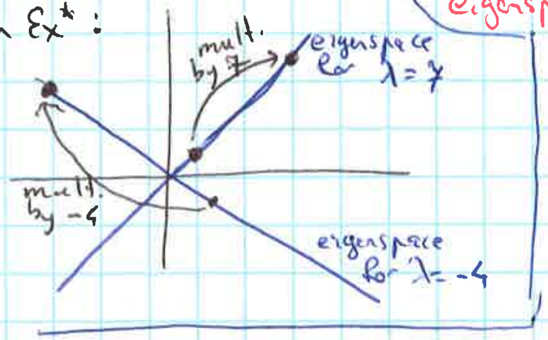
WARNING: We used row reduction to find eigenvectors but it cannot be used to find eigenvalues! REF of A does not display the eigenvalues of A!

for A $n\times n$, λ is an e.v. iff $\boxed{(A-\lambda I)\vec{x}=\vec{0}}$ $(**)$ has a nontriv. sol.

Set of solutions of $(**)$ $= \mathrm{Nul}(A-\lambda I) \subset \mathbb{R}^n$

eigenspace of A corresp. to λ $= \vec{0} \cup \{\text{all eigenvectors for } \lambda\}$

In Ex*:



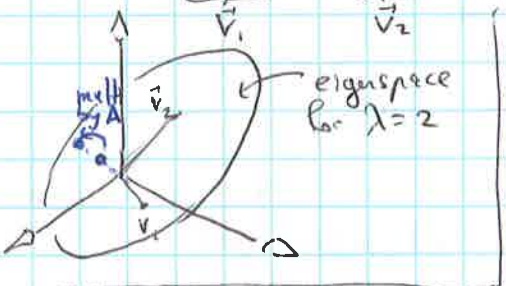
Ex: $A=\begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$, $\lambda=2$. Find a basis for the eigenspace H

Sol: $A-2I=\begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$

Aug. Mat. of $(**)$: $\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}}_{\vec{v}_1} + x_3 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}_2}$.

Thus H - plane in $\mathbb{R}^3$, $\{\vec{v}_1,\vec{v}_2\}$ - basis.



THM Eigenvalues of a triangular matrix are the diagonal entries.

Idea: $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$ $\quad A-\lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ 0 & a_{22}-\lambda & a_{23} \\ 0 & 0 & a_{33}-\lambda \end{bmatrix}$

λ e.v. $\iff \det(\ \ ) = 0 \iff \lambda \in \{a_{11}, a_{22}, a_{33}\}$

$\det(\ \ ) = (a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda)$ $\boxed{\checkmark}$

Ex: $A=\begin{bmatrix} 3 & 0 & 0 \\ 1 & 0 & 0 \\ -2 & -1 & 2 \end{bmatrix}$ - lower-triang.
$\lambda = 3, 0, 2$

Note: $\lambda=0$ is an eigenvalue $\iff$ $A\vec{x}=\vec{0}$ has a nontriv. sol. $\iff$ A non-invertible!

THM If $\{\vec{v}_1,\dots,\vec{v}_r\}$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1,\dots,\lambda_r$ of an $n\times n$ mat A, then the set $\{\vec{v}_1,\dots,\vec{v}_r\}$ is LI.

Application (discrete dynamical systems / difference equations)

$\vec{x}_{k+1} = \underbrace{A}_{n\times n}\vec{x}_k$, $k=0,1,2,\dots$ - recursive description of a sequence of vectors $\vec{x}_k$ in $\mathbb{R}^n$.

If $\vec{x}_0$ is an eigenvector corresp to λ, then $\vec{x}_k = \lambda^k \vec{x}_0$.
- and lin. comb. of such solutions are solutions too!

5.2. The characteristic equation

Ex: find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

Sol: λ e.v. $\Leftrightarrow$ $(A - \lambda I)\vec{x} = \vec{0}$ has a nontriv. sol. $\Leftrightarrow$ ~~det~~ $A - \lambda I$ not invertible $\Leftrightarrow$ $\boxed{\det(A - \lambda I) = 0}$

$A - \lambda I = \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$ $\quad \det = (2-\lambda)(-6-\lambda) - 3 \cdot 3 = \lambda^2 + 4\lambda - 21 = (\lambda - 3)(\lambda + 7)$
$= 0$ iff $\lambda \in \{3, -7\}$

Thus: $\boxed{\lambda = 3,\ \lambda = -7}$ are the eigenvalues

Inv. Mat. THM (Cont'd): A $n \times n$ mat is invertible iff
(s) 0 is not an eigenvalue of A $\quad$ (t) $\det A \neq 0$.

- λ is an eigenvalue of A iff λ satisfies the characteristic equation $\boxed{\det(A - \lambda I) = 0.}$

Ex: $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{bmatrix}$ Q: Find the char. eq.: $\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 & 2 \\ 0 & 1-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{vmatrix} = -(\lambda-3)^2(\lambda-1)$

So, char. eq.: $\underbrace{-(\lambda-3)^2(\lambda-1)}_{\text{characteristic polynomial of } A} = 0$

Note: $\lambda = 3$ - e.v. with (algebraic) multiplicity 2.
(multiplicity as a root of char. eq.)

Ex A 6×6, char. poly $= \lambda^6 - 4\lambda^5 - 12\lambda^4$ Q: find eigenvalues and their multiplicities

Sol: char poly $= \lambda^4(\lambda - 6)(\lambda + 2)$. So, eigenvals are $\lambda = 0$ (mult. 4)
$\lambda = 6$ (mult. 1), $\lambda = -2$ (mult. 1).

5.3

- for A $n \times n$, char. eq. has n roots (counting w/ multiplicities); some of them can be complex.

Similarity

Def A is similar to B if there is an invertible P s.t. $P^{-1}AP = B$ or equivalently $A = PBP^{-1}$

- $A \longmapsto P^{-1}AP$ - similarity transformation.

Note: $A \underset{\text{similar}}{\approx} B \Rightarrow B \approx A$

THM: If ~~nxn~~ matrices A and B are similar, they have the same char. polynomial and hence same eigenvalues (with same multiplicities)

Warning 1. $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \not\approx \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ - same eigenvalues but not similar

2. similarity is not the same as row equivalence! row operations change eigenvalues.

Study Ex. 5