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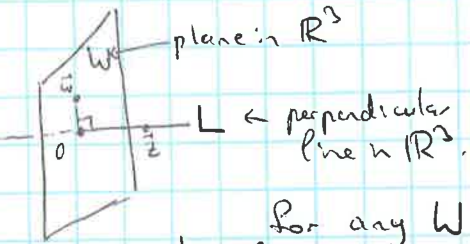
Orthogonal complements

- If $\vec{z} \in \mathbb{R}^n$ is orthogonal to every vector in $W \subset \mathbb{R}^n$ (a subspace), then $\vec{z}$ is orthogonal to W

Set of all vectors in $\mathbb{R}^n$ orthogonal to W - "orthogonal complement of W", $W^{\perp}$ - notation

Ex



$W^{\perp} = L$, $L^{\perp} = W$

for any W

- $\vec{x}$ is in $W^{\perp}$ iff $\vec{x}$ is orthog. to every vector in a set which spans W
- $W^{\perp}$ is a subspace of $\mathbb{R}^n$.

Ex for A $m \times n$ mat., $\quad$ Nul A and Row A $\subset \mathbb{R}^n$ - orthogonal complements of each other

Nul A^T and Col A $\subset \mathbb{R}^m$ - orthogonal complements of each other.

- for $W \subset \mathbb{R}^n$, $\boxed{\dim W + \dim W^{\perp} = n}$

6.2 Orthogonal sets

A set of vectors $\{\vec{u}_1, \dots, \vec{u}_p\}$ in $\mathbb{R}^n$ is an orthogonal set, if. $\vec{u}_i \cdot \vec{u}_j = 0$ for each pair $i \neq j$.

Ex*: $\vec{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix}$ $\quad$ Q: show that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthog. set

Sol: $\vec{u}_1 \cdot \vec{u}_2 = 3(-1) + 1 \cdot 2 + 1 \cdot 1 = 0$ ✓, $\vec{u}_1 \cdot \vec{u}_3 = 3(-\frac{1}{2}) + 1 \cdot (-2) + 1 \cdot \frac{7}{2} = 0$ ✓ $\quad \vec{u}_2 \cdot \vec{u}_3 = (-1)(-\frac{1}{2}) + 2(-2) + \frac{7}{2} = 0$ ✓

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THM If $S = \{\vec{u}_1, \dots, \vec{u}_p\}$ an orthog. set of nonzero vectors in $\mathbb{R}^n$, then S is lin. indep. and hence a basis for Span S.

- an orthogonal basis for a subspace $W \subset \mathbb{R}^n$ is a basis for W which is also an orthogonal set.

THM* Let $\{\vec{u}_1, \dots, \vec{u}_p\}$ be an orthog. basis for $W \subset \mathbb{R}^n$. For each $\vec{y} \in W$, weights in $\vec{y} = c_1 \vec{u}_1 + \dots + c_p \vec{u}_p$ are given by $\boxed{c_j = \dfrac{\vec{y} \cdot \vec{u}_j}{\vec{u}_j \cdot \vec{u}_j}}$, $j = 1..p$.

(Indeed: $\vec{y} \cdot \vec{u}_1 = c_1 \vec{u}_1 \cdot \vec{u}_1 + \cancel{c_2 \vec{u}_2 \cdot \vec{u}_1 + \dots + c_p \vec{u}_p \cdot \vec{u}_1} \Rightarrow c_1 = \dfrac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}$ and similarly for other c_j)

Ex: orthog. set $S = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ from Ex* is a basis for $\mathbb{R}^3$, $\vec{y} = \begin{bmatrix} 6 \\ 1 \\ -8 \end{bmatrix}$ $\quad$ Q: Express $\vec{y}$ as a lin. comb. of vectors in S

Sol: $\vec{y} = \underbrace{\dfrac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}}_{11} \vec{u}_1 + \underbrace{\dfrac{\vec{y} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2}}_{-12} \vec{u}_2 + \underbrace{\dfrac{\vec{y} \cdot \vec{u}_3}{\vec{u}_3 \cdot \vec{u}_3}}_{-33} \vec{u}_3 = \vec{u}_1 - 2\vec{u}_2 - 2\vec{u}_3$ ← did not need to solve the lin. system to compute the weights!

Orthogonal projection

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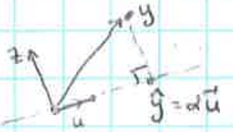


Given $\vec{u} \neq \vec{0}$ in $\mathbb{R}^n$, want to write $\vec{y} \in \mathbb{R}^n$

as $\vec{y} = \underbrace{\hat{y}}_{\alpha \vec{u} \text{ for some } \alpha} + \underbrace{\hat{z}}_{\text{orthog. to } \vec{u}}$ $\Rightarrow$ $(\vec{y} - \alpha\vec{u}) \cdot \vec{u} = 0$ $\Rightarrow$ $\alpha = \dfrac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}$

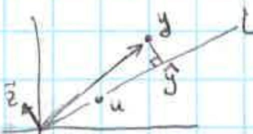
$\vec{y}\cdot\vec{u} = \alpha\,\vec{u}\cdot\vec{u}$

$$\hat{y} = \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} =: \text{proj}_L \vec{y}$$ - orthogonal projection of $\vec{y}$ onto $\underset{\text{the line}}{L} = \text{Span}\{\vec{u}\}$

does not change if $\vec{u} \mapsto c\vec{u}$, $c \neq 0$.

Ex: $\vec{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ $\vec{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Q: write $\vec{y}$ as $\underset{\text{in Span}\{\vec{u}\}}{\hat{y}} + \underset{\text{orthog to } \vec{u}}{\vec{z}}$

Sol: $\vec{y}\cdot\vec{u} = 40$ $\vec{u}\cdot\vec{u} = 20$ $\Rightarrow \hat{y} = \dfrac{40}{20}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$, $= 2\vec{u}$, $\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. So: $\underset{\vec{y}}{\begin{bmatrix} 7 \\ 6 \end{bmatrix}} = \underset{\hat{y}}{\begin{bmatrix} 8 \\ 4 \end{bmatrix}} + \underset{\vec{z}}{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}$



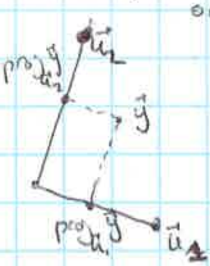
Q: find $\text{dist}(\vec{y}, L)$, $L = \text{Span}\{\vec{u}\}$

Sol: $\text{dist}(\vec{y}, L) = \text{dist}(\vec{y}, \hat{y}) = \|\vec{y} - \hat{y}\|$ — $\hat{y}$: closest point on L to $\vec{y}$

$= \|\vec{z}\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$

Ex: (THM*, geom. picture)
$W = \mathbb{R}^2 = \text{Span}\{\vec{u}_1, \vec{u}_2\}$ orthogonal

$\vec{y} \in \mathbb{R}^2$ can be written as $\vec{y} = \underbrace{\dfrac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1}_{\text{proj}_{\vec{u}_1}\vec{y}} + \underbrace{\dfrac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2}_{\text{proj}_{\vec{u}_2}\vec{y}}$



So: THM* decomposes $\vec{y}$ into a sum of orthog. projections onto 1-dim subspaces (which are mutually orthogonal)

Orthonormal sets $S = \{\vec{u}_1, \dots, \vec{u}_p\}$ is orthonormal if it is an orthog. set of unit vectors.

If $W = \text{Span}\,\underset{o/n}{S}$, then S - o/n basis for W.

Ex: $\{\vec{e}_1, \dots, \vec{e}_n\}$ is an o/n basis for $\mathbb{R}^n$.

Ex: $\vec{v}_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix}$ $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ - o/n basis for $\mathbb{R}^3$.

- obtained from Ex* by normalizing vectors to unit length: $\vec{v}_i = \dfrac{1}{\|\vec{u}_i\|}\vec{u}_i$

Thm: An $m \times n$ mat. U has o/n columns iff $U^T U = I$

Thm Let U be a $m \times n$ mat with o/n columns and let $\vec{x}, \vec{y} \in \mathbb{R}^n$. Then:
(a) $\|U\vec{x}\| = \|\vec{x}\|$ (b) $(U\vec{x})\cdot(U\vec{y}) = \vec{x}\cdot\vec{y}$ (c) $(U\vec{x})\cdot(U\vec{y}) = 0$ iff $\vec{x}\cdot\vec{y} = 0$

I.e. mapping $\vec{x} \mapsto U\vec{x}$ preserves length and orthogonality.

- Case $m = n$: square U with o/n columns is an orthogonal matrix. U orthogonal iff $U^{-1} = U^T$

Orthogonal projections

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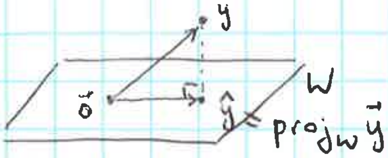
Given $\vec{y} \in \mathbb{R}^n$ and $W \subset \mathbb{R}^n$, there exists a unique ~~want to find~~ $\hat{\vec{y}} \in W$ s.t. (1) $\vec{y} - \hat{\vec{y}} \perp W$

(2) $\hat{\vec{y}}$ is the closest vector in W to $\vec{y}$



THM (The Orthogonal Decomposition THM)

Let W be a subspace of $\mathbb{R}^n$. Then each $\vec{y}$ in $\mathbb{R}^n$ can be written uniquely as $\vec{y} = \hat{\vec{y}} + \vec{z}$ with $\hat{\vec{y}} \in W$ and $\vec{z} \in W^\perp$. Moreover, if $\{\vec{u}_1, \dots, \vec{u}_p\}$ is any orthogonal basis for W, then $\boxed{\hat{\vec{y}} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \dots + \frac{\vec{y}\cdot\vec{u}_p}{\vec{u}_p\cdot\vec{u}_p}\vec{u}_p}$ (**) and $\vec{z} = \vec{y} - \hat{\vec{y}}$.

$\hat{\vec{y}} =: proj_W \vec{y}$ is the orthogonal projection of $\vec{y}$ onto W.

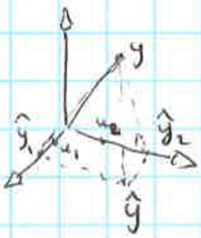
Ex@: $\underbrace{\vec{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}_{\text{orthog. basis for } W = Span\{\vec{u}_1, \vec{u}_2\}}$, $\vec{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Q: write $\vec{y}$ as a sum of a vector in W and a vector orthogonal to W.

Sol: $\hat{\vec{y}} = \frac{\vec{y}\cdot\vec{u}_1}{\vec{u}_1\cdot\vec{u}_1}\vec{u}_1 + \frac{\vec{y}\cdot\vec{u}_2}{\vec{u}_2\cdot\vec{u}_2}\vec{u}_2 = \underset{\frac{3}{10}}{\frac{9}{30}}\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \underset{\frac{5}{10}}{\frac{3}{6}}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}}_{\hat{y}}$; $\vec{z} = \vec{y} - \hat{\vec{y}} = \begin{bmatrix} 7/5 \\ 0 \\ 14/5 \end{bmatrix}$

So: $\vec{y} = \underset{\in W}{\begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}} + \underset{\in W^\perp}{\begin{bmatrix} 7/5 \\ 0 \\ 14/5 \end{bmatrix}}$

- formula (**) is the sum of projections of $\vec{y}$ onto lines $Span\{\vec{u}_1\}, \dots, Span\{\vec{u}_p\}$



- If $\vec{y} \in W$, $\boxed{proj_W \vec{y} = \vec{y}}$

THM (best approximation theorem)

Let $W \subset \mathbb{R}^n$, $\vec{y} \in \mathbb{R}^n$ and $\hat{\vec{y}} = proj_W \vec{y}$. Then $\hat{\vec{y}}$ is the closest point in W to $\vec{y}$, i.e., $\|\vec{y} - \hat{\vec{y}}\| < \|\vec{y} - \vec{v}\|$ for all $\vec{v} \in W$, $\vec{v} \neq \hat{\vec{y}}$.

$\hat{\vec{y}}$ is the best approximation of $\vec{y}$ by elements of W, $\|\vec{y} - \hat{\vec{y}}\|$ - "error" of the approximation.

Ex@: $\hat{\vec{y}} = \begin{bmatrix} -2/5 \\ 2 \\ 1/5 \end{bmatrix}$ - closest point to $\vec{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ on the plane W.

$\underbrace{dist(\vec{y}, W)}_{\text{dist. between } \vec{y} \text{ and closest point to } \vec{y} \text{ on } W} = \|\vec{y} - \hat{\vec{y}}\| = \|\vec{z}\| = \frac{7}{5}\sqrt{5} = \frac{7}{\sqrt{5}}$

THM If $\{\vec{u}_1, \dots, \vec{u}_p\}$ is an orthonormal basis for $W \subset \mathbb{R}^n$, then

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$\text{proj}_W \vec{y} = (y \cdot \vec{u}_1)\vec{u}_1 + \dots + (\vec{y} \cdot \vec{u}_p)\vec{u}_p$

If $U = [\vec{u}_1 \; \vec{u}_2 \cdots \vec{u}_p]$, then $\boxed{\text{proj}_W \vec{y} = UU^T \vec{y}}$ for all $\vec{y} \in \mathbb{R}^n$.