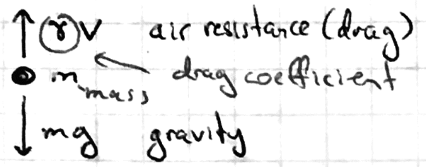
Differential equations (Boyce, di Prima)

3/28/2018

1.1. Direction fields

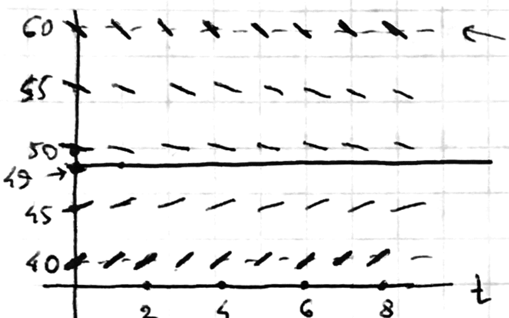
Ex 1 falling object

quantity of interest $v(t)$ — velocity in m/s; time in s



$$F = m\frac{dv}{dt} \rightsquigarrow \boxed{m\frac{dv}{dt} = mg - \gamma v} \quad (**)$$

Set $m = 10$ kg, $\gamma = 2$ kg/s. Diff. eq. becomes: $\boxed{\boxed{\frac{dv}{dt} = 9.8 - \frac{v}{5}}}$ $(*)$



← for $v=60$, $\frac{dv}{dt} = -2.2$

← for $\boxed{v=49}$, $\frac{dv}{dt} = 0$ - equilibrium solution

if $v = 40$, $\frac{dv}{dt} = 1.8$ ← slope of the solution

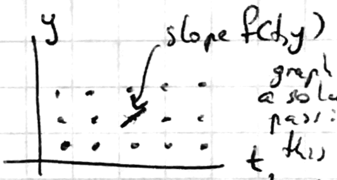
for $v > 49$ slopes are negative (speed decreasing)

for $v < 49$ slopes are positive (speed increasing)

solutions converge to $v = 49$ m/s (or generally, $v = \frac{mg}{\gamma}$ in $(**)$) as $t \to \infty$.

↖ terminal velocity

- For diff.eq. of form $\frac{dy}{dt} = \underset{\text{"rate function"}}{f(t,y)}$, can form a direction field



graph of a solution passing through this point has to have this slope.

Ex 2 Field mice and owls

$p(t)$ ← time

population of mice in an area

$$\frac{dp}{dt} = \boxed{rp}\ \boxed{-k}$$

growth proportional to current population (hypothesis)

rate constant (growth rate)

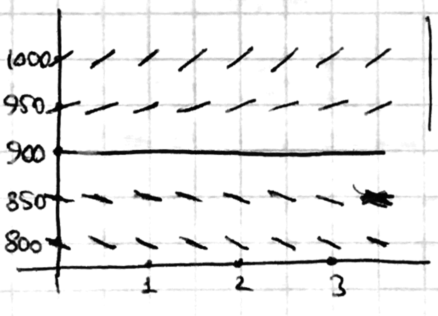
mice killed by owls per unit of time

predator term

E.g.: $\boxed{\frac{dp}{dt} = \frac{p}{2} - 450}$ in months

$r = 0.5$

$k = 15/\text{day}$
$= 450/\text{month}$



$\frac{dp}{dt} > 0$ (population increases)

$p = 900$ - equilibrium solution

$\frac{dp}{dt} < 0$ (population decreases)

solutions diverge from (are repelled by) the equilibrium solution.

- (solutions starting below 900 will eventually become negative - limitation of the model)
  - unrealistic

$>900 \to p$ becomes huge soon

1.2 Solutions of some diff. eq.

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Ex 2 (cont'd) $\boxed{\frac{dp}{dt} = 0.5p - 450}$ $(**)$ rewrite $\frac{dp}{dt} = \frac{p-900}{2}$ or $\frac{dp/dt}{p-900} = \frac{1}{2}$ (@)

Note: $\frac{d}{dt} \ln|p-900| = \underbrace{\frac{1}{p-900}}_{\frac{d \ln|p-900|}{dp}} \frac{dp}{dt}$ = - l.h.s. of the eq. (@)

chain rule

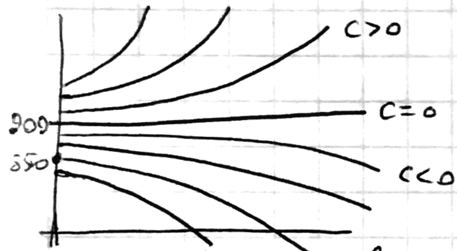
So (@) reads $\frac{d}{dt} \ln|p-900| = \frac{1}{2} \xrightarrow[\text{both sides}]{\text{integrate}} \ln|p-900| = \frac{t}{2} + \underbrace{C}_{}$ $\longrightarrow$

arbitrary constant of integration

$\xrightarrow[\text{exp}]{} |p-900| = e^C e^{t/2}$ or $p - 900 = \underbrace{\pm e^C}_{c} e^{t/2}$

or $\boxed{p = 900 + c e^{t/2}}$ (@@), c arbitrary (nonzero) constant.

Note: p = 900 also a solution ~~of (@)~~ of (**), contained in solution (@@) if we allow c = 0.



graphs of solutions for different values of C.

- we found an infinite family of solutions, one per value of C.

Initial value problem:

$\frac{dp}{dt} = 0.5p - 450$, $p(0) = 850$ ← initial condition

from (@@) general sol., $p(0) = 900 + C = 850 \Rightarrow C = -50$. Thus sol: $\boxed{p(t) = 850 - 50e^{t/2}}$

- more generally: $\frac{dy}{dt} = ay - b$, $y(0) = y_0$ - init. cond.

diff. eq.: $\frac{dy/dt}{y - b/a} = a \Leftrightarrow \frac{d}{dt} \ln\left|y - \frac{b}{a}\right| = a \xrightarrow[\text{integrate}]{} \ln\left|y - \frac{b}{a}\right| = at + C \rightsquigarrow \boxed{y(t) = \frac{b}{a} + ⓒ e^{at}}$ ← General Solution

arbitrary constant
C=0 - equilibrium solution

to satisfy init. cond.: $y(0) = \frac{b}{a} + c = y_0 \Rightarrow c = y_0 - \frac{b}{a}$

solve for c $\Rightarrow \boxed{y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right)e^{at}}$ ← solution of the init. val. problem

general sol. produces a family of curves on (t, y) plane - "integral curves"

init. val. prob. - finding the integral curve passing through a given initial point.

Ex 1 (cont'd) $m\frac{dv}{dt} = mg - \gamma v$, $v(0) = v_0$ $a = -\frac{\gamma}{m}$ $b = -g$, $y(t) = v(t)$ $\rightsquigarrow v(t) = \boxed{\frac{mg}{\gamma}} + \left(v_0 - \frac{mg}{\gamma}\right)e^{-\gamma t/m}$

terminal velocity

initial velocity

$\frac{03/28/2018}{3}$

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$

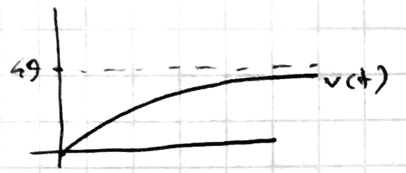
assume the object is dropped from a height of 300m.

Q: find its velocity at time t

- how long will it take to fall to the ground?
- velocity at t of impact?

Sol: "dropped" implies $v(0)=0$ => sol. $v(t) = 49 + ce^{-t/5}$

with $v(0) = 0 \Rightarrow c = -49 \Rightarrow \boxed{v(t) = 49(1-e^{-t/5})}$



distance has the object fallen

$$\frac{dx}{dt} = v(t) = 49(1-e^{-t/5})$$

$$\xrightarrow[\text{integrate}]{} x = 49t + 245e^{-t/5} + \underbrace{k}_{\text{const. of integration}}$$

$x(0)=0 \Rightarrow k = -245 \Rightarrow x(t) = 49t + 245e^{-t/5} - 245$

$$\underset{\substack{\uparrow \\ \text{time of impact}}}{x(T)} = 49T + 245e^{-T/5} - 245 = 300 \xrightarrow[\text{numerically}]{\text{solve}} T \approx 10.51\text{ s}$$

$v(T) \approx 43.01$ m/s from