5.2. The characteristic equation

03/02/2018
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Ex: find the eigenvalues of $A=\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$

Sol: $(A-\lambda I)\vec{x}=\vec{0}$ has a nontriv. sol. $\iff$ ~~det~~ $A-\lambda I$ not invertible $\iff$ $\boxed{\det(A-\lambda I)=0}$

λ e.v.

$$A-\lambda I=\begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix} \qquad \det = (2-\lambda)(-6-\lambda)-3\cdot 3 = \lambda^2+4\lambda-21 = (\lambda-3)(\lambda+7)$$

$=0$ iff $\lambda\in\{3,-7\}$

Thus: $\boxed{\lambda=3,\ \lambda=-7}$ are the eigenvalues

Inv. Mat. THM (Cont'd): A $n\times n$ mat is invertible iff

(s) 0 is not an eigenvalue of A (t) $\det A\neq 0$

- λ is an eigenvalue of A iff λ satisfies the characteristic equation $\boxed{\det(A-\lambda I)=0.}$

Ex: $A=\begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{bmatrix}$ Q: Find the char. eq.: $\det(A-\lambda I)=\begin{vmatrix} 3-\lambda & 1 & 2 \\ 0 & 1-\lambda & 5 \\ 0 & 0 & 3-\lambda \end{vmatrix} = -(\lambda-3)^2(\lambda-1)$

So, char. eq.: $\underbrace{-(\lambda-3)^2(\lambda-1)}_{\text{characteristic polynomial of } A}=0$

Note: $\lambda=3$ - e.v. with (algebraic) multiplicity 2.
(multiplicity as a root of char. eq.)

Ex A 6×6, char. poly $=\lambda^6-4\lambda^5-12\lambda^4$ Q: find eigenvalues and their multiplicities

Sol: char poly $=\lambda^4(\lambda-6)(\lambda+2)$. So, eigenvals are $\lambda=0$ (mult. 4)
$\lambda=6$ (mult. 1), $\lambda=-2$ (mult. 1).

5.3

03/05/2018
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- for A $n\times n$, char. eq. has n roots (counting w/ multiplicities); some of them can be complex.

Similarity

Def A is similar to B if there is an invertible P s.t. $P^{-1}AP=B$ or equivalently $A=PBP^{-1}$

- $A\longmapsto P^{-1}AP$ - similarity transformation. Note: $A\underset{\text{similar}}{\approx}B \Rightarrow B\approx A$

THM: If ~~[illegible]~~ matrices A and B are similar, they have the same char. polynomial and hence same eigenvalues (with same multiplicities)

Warning 1. $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\not\approx\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ - same eigenvalues but not similar

2. similarity is not the same as row equivalence! row operations change eigenvalues.

Study Ex. 5

5.3 Diagonalization

05/05/2018
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Often we can factorize $A = PDP^{-1}$ ← diagonal matrix (the D) — allows to compute A^k efficiently for large k.

Ex: • $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$, $D^2 = \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}$, $D^3 = \begin{bmatrix} 5^3 & 0 \\ 0 & 3^3 \end{bmatrix}$, ... $D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}$

• $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} = PDP^{-1}$, $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$. Then: $A^2 = PD\cancel{P^{-1}P}DP^{-1} = PD^2P^{-1}$

$A^3 = (PD\cancel{P^{-1}})(\cancel{P}D^2P^{-1}) = PD^3P^{-1}$

$A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$

DEF

A is <u>diagonalizable</u> if $A \approx D$ - diag mat.
i.e. if $A = PDP^{-1}$ for some diagonal D and invertible P.

THM (The diagonalization THM)
An $n \times n$ mat. A is <u>diagonalizable</u> iff A has n <u>LI</u> eigenvectors $\vec{v}_1, \dots, \vec{v}_n$

$\lambda_1 \dots \lambda_n$ - corresp. eigenvalues.

In fact, $A = PDP^{-1}$, $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & & \lambda_n \end{bmatrix}$, $P = [\vec{v}_1 \dots \vec{v}_n]$

Ex: $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ Q: diagonalize if possible
i.e. find P, D s.t. $A = PDP^{-1}$.

Sol Step I. find the eigenvalues of A. Char. eq: $0 = \det(A - \lambda I) = -(\lambda - 1)(\lambda + 2)^2$

Sol: $\lambda = 1, -2$ - eigenvalues

Step II Find 3 LI eigenvectors. [If it fails, A cannot be diagonalized!]
because A is 3×3

basis for $\lambda = 1$ eigenspace: $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

basis for $\lambda = -2$: $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ - LI set.

Step III Construct $P = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

Step IV Construct $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ Warning: order of λ's in D should match the order of v's in P.

Check: $\underbrace{AP = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}, \quad PD = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}}_{\text{hooray!}}$

Ex: $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ Q: Diagonalizable?

Sol: $\lambda = 3$ - the only eigenvalue. Basis for $\lambda = 3$ eigenspace = basis for $\mathrm{Nul}\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A-3I} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

- cannot find two LI eigenvectors => A not diagonalizable!

THM An $n \times n$ mat. A with n distinct eigenvalues is diagonalizable

Ex: $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 5 \\ 0 & 0 & 2 \end{bmatrix}$ Diagonalizable?

Case of Non-distinct eigenvalues

THM Let A be $n \times n$ mat. ~~with eigen~~ whose distinct eigenvalues are $\lambda_1, \dots, \lambda_p$
$m_1, \dots, m_p$ - multiplicities

(a) for each $1 \le k \le p$, dimension d_k of the eigenspace H_k for λ_k is $\le m_k$.

(b) A is diagonalizable iff $\boxed{\sum_k d_k = n}$, i.e. iff (i) char. poly. factors completely into linear factors
(ii) $d_k = m_k$ for each k.

(c) if A is diagonalizable and B_k - basis for eigenspace H_k. Then $B_1 \cup B_2 \cup \dots \cup B_p$ - eigenvector basis for $\mathbb{R}^n$.

Ex: [Ex 6] $A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$ Q: diagonalize if possible.

Sol: $\lambda = 5, -3$ eigenvalues
2, 2 - multiplicities

basis for $\lambda = 5$: $\vec{v}_1 = \begin{bmatrix} -8 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -16 \\ 4 \\ 0 \\ 1 \end{bmatrix}$

basis for $\lambda = -3$: $\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $\vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

by THM, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ LI, so:

$A = PDP^{-1}$, $P = \begin{bmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$