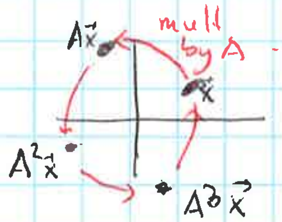
5.5. Complex eigenvalues.

03/09/2018

Ex $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$



A has no eigenvectors in $\mathbb{R}^2$!

char. eq. : $\lambda^2 + 1 = 0$ complex roots: $\lambda = i$, $\lambda = -i$. If we allow A to act on $\underline{\underline{\mathbb{C}^2}}$:

$A \underbrace{\begin{bmatrix} 1 \\ -i \end{bmatrix}}_{\text{eigenvector for } \lambda = i} = \begin{bmatrix} i \\ 1 \end{bmatrix} = i \begin{bmatrix} 1 \\ -i \end{bmatrix}$ $\qquad$ $A \underbrace{\begin{bmatrix} 1 \\ i \end{bmatrix}}_{\text{eigenvector for } \lambda = -i} = \begin{bmatrix} -i \\ 1 \end{bmatrix} = -i \begin{bmatrix} 1 \\ i \end{bmatrix}$

Ex* $A = \begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix}$ Q: find eigenvalues and a basis for each eigenspace.

Sol. char. eq. $0 = \begin{vmatrix} 0.5-\lambda & -0.6 \\ 0.75 & 1.1-\lambda \end{vmatrix} = \lambda^2 - 1.6\lambda + 1$ solutions: $\lambda = \dfrac{1.6 \pm \sqrt{\overbrace{(-1.6)^2 - 4}^{2.56-4=-1.44}}}{2}$

$= \boxed{0.8 \pm 0.6i}$

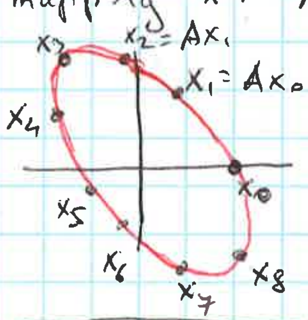
for the eigenvalue $\lambda = 0.8 - 0.6i$: $A - (0.8 - 0.6i)I = \begin{bmatrix} -0.3+0.6i & -0.6 \\ 0.75 & 0.3+0.6i \end{bmatrix}$

matrix eq. for the eigenvector:
$\left.\begin{aligned} &(1)\ (-0.3+0.6i)x_1 - 0.6x_2 = 0 \\ &(2)\ 0.75x_1 + (0.3+0.6i)x_2 = 0 \end{aligned}\right\}$ eq. has a nontriv. sol => both equations determine same rel. between x_1, x_2

$(1) \Leftrightarrow (2) \iff x_1 = -(0.4 + 0.8i)x_2$
choose $x_2 = 5$ => basis for the eigenspace: $\vec{v}_1 = \begin{bmatrix} -2-4i \\ 5 \end{bmatrix}$

analogously, for $\lambda = 0.8 + 0.6i$, eigenvector: $\vec{v}_2 = \begin{bmatrix} -2+4i \\ 5 \end{bmatrix}$

- mapping $\vec{x} \mapsto A\vec{x}$, with A above, is "essentially" a rotation



Real and Imaginary parts of vectors

for $\vec{x} \in \mathbb{C}^n$, complex conjugate $\overline{\vec{x}} \in \mathbb{C}^n$ - vector whose entries are complex conjugates of entries of $\vec{x}$.

Also: $\underbrace{\text{Re}\,\vec{x}}_{\text{real part of } \vec{x}}$ - vector of real parts of entries of $\vec{x}$, $\quad \text{Im}\,\vec{x}$ - likewise.

Ex: $\vec{x} = \begin{bmatrix} 3-i \\ i \\ 2+5i \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} + i \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$. Then: $\text{Re}\,\vec{x} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$, $\text{Im}\,\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix}$, $\overline{\vec{x}} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} - i \begin{bmatrix} -1 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 3+i \\ -i \\ 2-5i \end{bmatrix}$

for B $m \times n$ mat. with (possibly) complex entries, $\overline{B}$ - matrix with conjugate entries. We have: $\overline{r\vec{x}} = \overline{r}\,\overline{\vec{x}}$, $\overline{B\vec{x}} = \overline{B}\,\overline{\vec{x}}$, $\overline{BC} = \overline{B}\,\overline{C}$, $\overline{rB} = \overline{r}\,\overline{B}$

Eigenvalues and eigenvectors of a real matrix that acts on $\mathbb{C}^n$ (03/09/2018 2)

A $n\times n$ with real entries. Then $\overline{A\vec{x}} = \overline{A}\,\overline{\vec{x}} = A\overline{\vec{x}}$

If λ is an eigenvalue and $\vec{x}\in\mathbb{C}^n$ corresp. eigenvector, then $A\overline{\vec{x}} = \overline{A\vec{x}} = \overline{\lambda\vec{x}} = \overline{\lambda}\,\overline{\vec{x}}$

Thus: $\overline{\lambda}$ is also an eigenvalue with $\overline{\vec{x}}$ the corresp. eigenvector!

I.e. for A real, its complex eigenvalues occur in conjugate pairs!

$$\lambda = a+ib, \; b\neq 0.$$

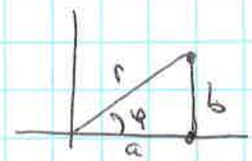
Ex: for Ex*: $\lambda = 0.8-0.6i$ $\quad \overline{\lambda} = 0.8+0.6i$ - conjugate

$\vec{v}_1 = \begin{bmatrix} -2-4i \\ 5 \end{bmatrix}$ $\quad \vec{v}_2 = \begin{bmatrix} -2+4i \\ 5\end{bmatrix} = \overline{\vec{v}_1}$ - conjugate

Ex (building block for 2×2 matrices w/ cx eigenvalues)

for $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ with a,b real, nonzero, eigenvalues: $\lambda = a\pm ib$ and

$$C = \underset{|\lambda| = \sqrt{a^2+b^2}}{\underset{\shortparallel}{\underline{r}}} \begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix} = \underbrace{\begin{bmatrix} r & 0 \\ 0 & r\end{bmatrix}}_{\substack{\text{scaling} \\ \text{by } |\lambda|}} \cdot \underbrace{\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}}_{\substack{\text{rotation by } \varphi \\ \text{- argument of } \lambda = a+ib}}$$



Ex: (back to Ex*) Let $A = \begin{bmatrix} 0.5 & -0.6 \\ 0.75 & 1.1 \end{bmatrix}$, $\lambda = 0.8-0.6i$, $\vec{v}_1 = \begin{bmatrix} -2-4i \\ 5\end{bmatrix}$

Let $P = [\operatorname{Re} v_1 \; \operatorname{Im} v_1] = \begin{bmatrix} -2 & -4 \\ 5 & 0 \end{bmatrix}$ And let

$C = P^{-1}AP = \ldots = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}$ - pure rotation! (by $\varphi = \arctan\frac{0.6}{0.8}$) since $|\lambda| = \sqrt{(0.8)^2+(0.6)^2} = 1$

Thus $A = P\underset{\text{rotation}}{\underline{C}}P^{-1}$

$\vec{x} = P\vec{u}$ change of variable

$$\begin{tikzcd} \vec{x} \arrow[r, "A"] \arrow[d, "P^{-1}"', "\text{change of variable}"'] & A\vec{x} \\ \vec{u} \arrow[r, "C"', "\text{rotation}"] & C\vec{u} \arrow[u, "P"', "\text{change of variable}"] \end{tikzcd}$$

THM Let A be a real 2×2 mat. with a complex eigenvalue $\lambda = a-ib$, $b\neq 0$ and $\vec{v}$ the corresp. eigenvector in $\mathbb{C}^2$. Then:

$A = PCP^{-1}$ where $P = [\operatorname{Re}\vec{v} \;\; \operatorname{Im}\vec{v}]$, $C = \begin{bmatrix} a & -b \\ b & a\end{bmatrix}$

- Look at Ex 8 (sec. 5.5)