Autonomous differential equations and population dynamics

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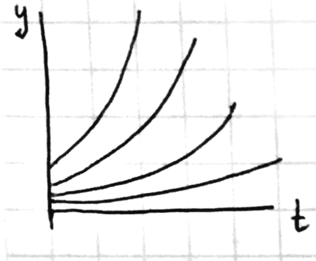
$\frac{dy}{dt} = f(y)$ – autonomous eq. (1st order ODE with t not appearing explicitly)

- Exponential growth $\frac{dy}{dt} = ry$, $y(0) = y_0$ $\longrightarrow$ $\boxed{y = y_0 e^{rt}}$

r: rate of growth (if $r>0$; rate of decline if $r<0$)

y: population of a species at time t

population grows exponentially with time



- can be accurate under ideal conditions, for a limited period of time

- Logistic growth. Idea: growth rate depends on present population. $\frac{dy}{dt} = h(y)\,y$

$h(y) \approx r > 0$ for y small, $h(y) < 0$ for y sufficiently large

model: $h = r - ay$, i.e.

$\boxed{\frac{dy}{dt} = (r - ay)y}$ – Verhulst eq. for logistic growth

or $\boxed{\frac{dy}{dt} = \underbrace{r\left(1 - \frac{y}{K}\right)y}_{f(y)}}$ $(*)$, $K = \frac{r}{a}$

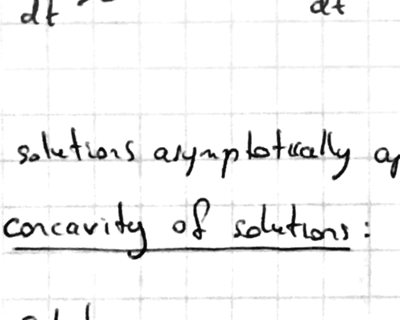
r: intrinsic growth rate

(constant) equilibrium solutions:

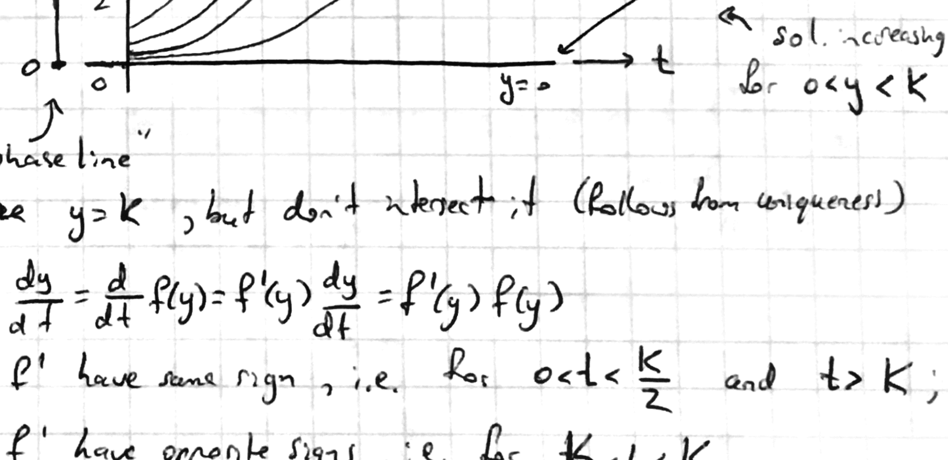
$y = 0 = \varphi_1(t)$

$y = K = \varphi_2(t)$

zeros of $f(y)$ ("critical points")



qualitative sketch of solutions:

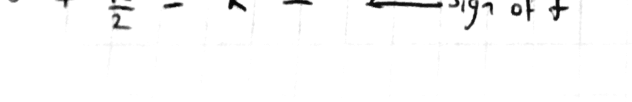


solutions asymptotically approach line $y = K$, but don't intersect it (follows from uniqueness)

Concavity of solutions: $\frac{d^2y}{dt^2} = \frac{d}{dt}\frac{dy}{dt} = \frac{d}{dt}f(y) = f'(y)\frac{dy}{dt} = f'(y)f(y)$

solution concave up if f and f' have same sign, i.e. for $0 < t < \frac{K}{2}$ and $t > K$;

concave down if f and f' have opposite signs, i.e. for $\frac{K}{2} < t < K$



inflection points on a solution occurs when $f'(y) = 0$, i.e. when $y = \frac{K}{2}$

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K is approached but never exceeded if $y_0 < K$
$\to$ K is the "saturation level" or "environmental carrying capacity"

Note: a non-linear term in eq. (*) created a drastically different behavior of solutions than in linear case

Explicit solution

$$\frac{dy}{(1-\frac{y}{K})y} = r\,dt \rightsquigarrow \left(\frac{1}{y} + \frac{1/K}{1-y/K}\right)dy = r\,dt \to \ln|y| - \ln\left|1-\frac{y}{K}\right| = rt + c$$

$$\to \frac{y}{1-y/K} = Ce^{rt} \to \boxed{y = \frac{y_0 K}{y_0 + (K-y_0)e^{-rt}}}$$

if $y_0 = 0$ then $y(t) = 0$
if $y_0 > 0$ then $y(t) \xrightarrow[t\to\infty]{} K$

$\lim_{t\to\infty} y(t) = K$

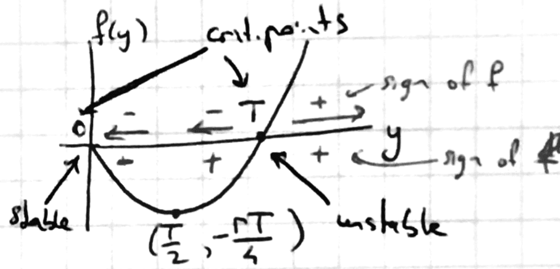
for each $y_0 > 0$, solution approaches equilibrium sol. $y = K$ - asymptotically stable solution

$y = 0$ - unstable equilibrium solution
- the only way to guarantee that sol. remains near zero is to make sure $y_0 = 0$ exactly.

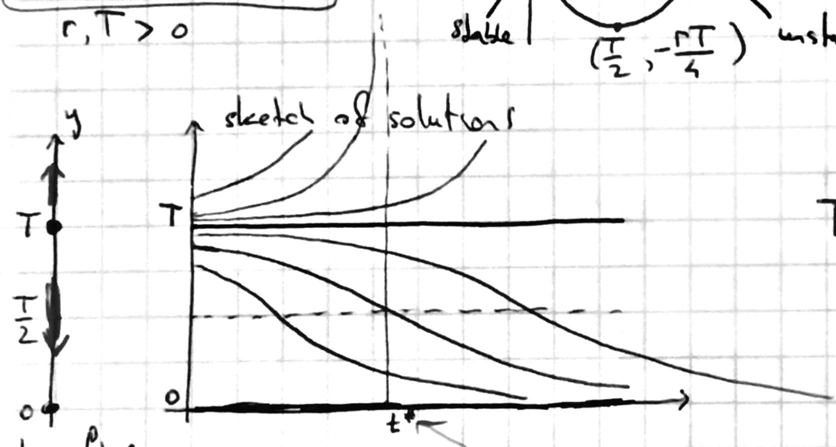
Critical threshold

$$\boxed{\frac{dy}{dt} = -r\left(1-\frac{y}{T}\right)y}$$

$r, T > 0$



concave up for $y < \frac{T}{2}$, $y > T$
concave down: $\frac{T}{2} < y < T$
inflection pts: $y = \frac{T}{2}$



T - threshold level, below which the growth does not occur

for $y_0 < T$, $\lim_{t\to\infty} y(t) = 0$

explicit sol.: $y = \frac{y_0 T}{y_0 + (T-y_0)e^{rt}}$.

If $y_0 > T$, denominator becomes zero at $t = t^* = \frac{1}{r}\ln\frac{y_0}{y_0 - T}$

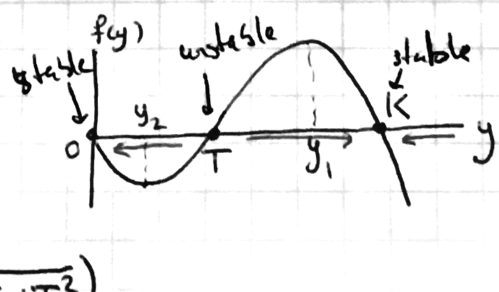
root of $y_0 - (y_0 - T)e^{rt^*} = 0$

$\Rightarrow$ Solution has a vertical asymptote at $t = t^*$

Logistic growth with a threshold

$$\frac{dy}{dt} = -r\left(1-\frac{y}{K}\right)\left(1-\frac{y}{T}\right)y$$

$r > 0$, $0 < T < K$



$y_{1,2} = \frac{1}{3}\left(K + T \pm \sqrt{K^2 - KT + T^2}\right)$
- roots of $f'(y) = 0$