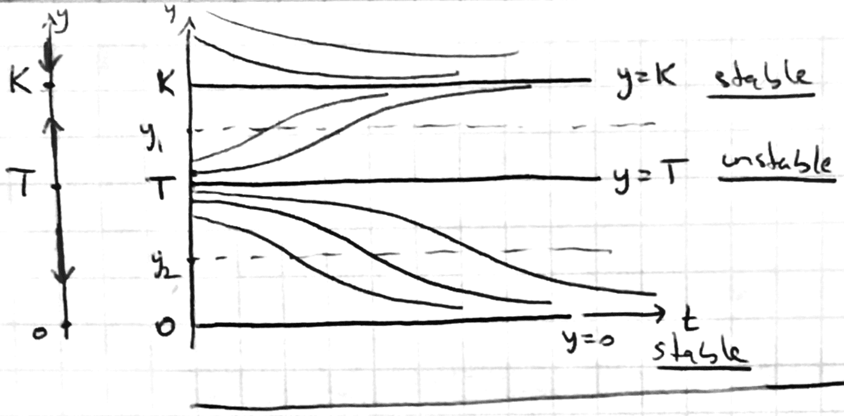

2.6. Exact differential equations and integrating factors

Ex: $\underbrace{2x+y^2}_{M} + \underbrace{2xy}_{N}\, y' = 0 \quad (*)$ – not linear, not separable (recalls separable case: $M(x) + N(y)y' = 0$)

Solution: ~~note~~ set $\psi(x,y) = x^2 + xy^2$ note: $\frac{\partial \psi}{\partial x} = 2x + y^2$; $\frac{\partial \psi}{\partial y} = 2xy$ $(**)$

I.e. $(*)$ is $\underbrace{\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}}_{} = 0$

$= \frac{d}{dx}\psi(x,y(x))$ by chain rule

$(*) \rightsquigarrow \frac{d}{dx}(x^2 + x\underset{\substack{\uparrow \\ \text{where} \\ y(x)}}{y^2}) = 0 \xrightarrow[\text{integrate}]{} \psi(x,y) = \boxed{x^2 + xy^2 = C}$

↑ implicitly defines solutions of $(*)$.

Level curves of ψ are integral curves of $(*)$

Note: finding ψ satisfying $(**)$ was instrumental for the solution

~~Generally~~

Generally: $M(x,y) + N(x,y)\, y' = 0 \quad (***)$

Suppose we can find $\psi(x,y)$ s.t. $\frac{\partial \psi}{\partial x} = M(x,y)$, $\frac{\partial \psi}{\partial y} = N(x,y)$ $(**)$ – Then (if such ψ exists), the eq. $(***)$ is called **exact**.

Then eq. $\Leftrightarrow$ $0 = M + Ny' = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} \underset{\substack{\uparrow \\ \text{chain rule}}}{=} \frac{d}{dx}\psi(x,y(x))$

Solutions are given implicitly by $\boxed{\psi(x,y) = C}$

- How to see whether such ψ exists and if so, how to find it?

THM ~~Let fuel~~ Assume $M, N, \underset{\frac{\partial}{\partial y}M}{M_y}, \underset{\frac{\partial}{\partial x}N}{N_x}$ continuous in a rectangle R: $\alpha < x < \beta$, $\gamma < y < \delta$.

Then eq $(***)$ is exact (in a rectangle R) iff $\boxed{M_y(x,y) = N_x(x,y)}$ $(\#)$

I.e., there exists ψ satisfying $\psi_x = M$, $\psi_y = N$ iff $M_y = N_x$

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Idea: (#) is necessary for exactness : $M_y = \frac{\partial}{\partial y}\underbrace{\left(\frac{\partial \psi}{\partial x}\right)}_{M} = \frac{\partial}{\partial x}\underbrace{\left(\frac{\partial \psi}{\partial y}\right)}_{N} = N_x$

how to find ψ: (1) $\psi_x = M$ ⟶ integrate in x (for fixed y); result will contain a "constant" $h(y)$.

(2) $\psi_y = N$ — substitute the result in (2) - find $h(y)$ from it.

Ex: $\underbrace{(y\cos x + 2xe^y)}_{M} + \underbrace{(\sin x + x^2e^y - 1)}_{N}y' = 0$

Sol: $M_y = \cos x + 2xe^y = N_x$ ⟹ eq. is exact!

WANT $\psi(x,y)$ s.t. $\psi_x = y\cos x + 2xe^y \xrightarrow{\text{integrate wrt } x} \psi = y\sin x + x^2e^y + h(y)$

$\psi_y = \sin x + x^2 e^y - 1$ ⟷ $\psi_y = \sin x + x^2e^y + h'(y)$

so: need $h'(y) = -1$ ⟶ take $h = -y$

Thus: $\psi = y\sin x + x^2e^y - y$ and solutions are given implicitly by $\boxed{y\sin x + x^2e^y - y = C}$

Ex: $\underbrace{(3xy + y^2)}_{M} + \underbrace{(x^2 + xy)}_{N}y' = 0$

Sol:

$M_y = 3x + 2y \neq N_x = 2x + y$ ⟶ eq. not exact!

still try: $\psi_x = 3xy + y^2 \xrightarrow{\text{integrate}} \psi = \frac{3}{2}x^2y + xy^2 + h(y)$ ⟶ $\psi_y = \frac{3}{2}x^2 + 2xy + h'(y)$

$\psi_y = x^2 + xy$ (##) $\xrightarrow{(\#\#)} h'(y) = -\frac{1}{2}x^2 - xy$ — impossible to solve! since rhs depends on x and lhs does not.

Integrating factors. IDEA: Try to convert an eq. to an exact one by multiplying by $\mu(x,y)$.

$M(x,y) + N(x,y)y' = 0 \xrightarrow[\cdot\mu(x,y)]{} \mu M + \mu N y' = 0$ (&) exact iff $(\mu M)_y = (\mu N)_x$ (@)

⟺ $\mu_y M + \mu M_y = \mu_x N + \mu N_x$ — PDE on μ - hard!

Consider the case $\mu = \mu(x)$ - indep. of y.

Then (@) becomes $\mu \cdot M_y = \mu_x N + \mu N_x$, i.e., $\frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu$ (@@)

So: if $\frac{M_y - N_x}{N}$ depends only on x (not on y), we can solve (@@) for $\mu(x)$ and then solve (&) as an exact eq!

Ex: $\underbrace{(3xy+y^2)}_{M}+\underbrace{(x^2+xy)}_{N}y'=0$ Q: find μ and solve the eq.

Sol: $\frac{M_y-N_x}{N}=\frac{3x+2y-2x-y}{x^2+xy}=\frac{x+y}{x^2+xy}=\frac{1}{x}$ ⇒ can find $\mu(x)$ from ~~dμ/dx = 1/x~~ $\frac{d\mu}{dx}=\frac{1}{x}\mu$

- indep. of y

⟶ $\boxed{\mu(x)=x}$ integrating factor

(eq)$\cdot\mu$: $\underbrace{(3x^2y+xy^2)}_{\tilde{M}}+\underbrace{(x^3+x^2y)}_{\tilde{N}}y'=0$

$\tilde{M}_y=3x^2+2xy=\tilde{N}_x$

$\psi_x=3x^2y+xy^2 \xrightarrow{\text{integrate in } x} \psi=x^3y+\frac{1}{2}x^2y^2+h(y)$

$\psi_y=x^3+x^2y$ $\psi_y=x^3+x^2y+h'(y)$

so: $h'(y)=0$, can take $h=0$

⇒ solutions given implicitly by $\boxed{x^3y+\frac{1}{2}x^2y^2=C}$

Remark: there is a second integrating factor $\mu(x,y)=\frac{1}{xy(2x+y)}$ - depends on x and y
- harder to find.