3.2 Solutions of linear homogeneous equations. Wronskian

THM (Existence & uniqueness)

$$\boxed{\begin{gathered} y''+p(t)y'+q(t)y=g(t) \\ y(t_0)=y_0,\ y'(t_0)=y_0' \end{gathered}}$$

Assume p, q, g continuous on an interval $\alpha<t<\beta$ containing t_0. Then the IVP has exactly one solution, and the sol. exists for $\alpha<t<\beta$.

I.e. we have: • existence • uniqueness • Sol. exists throughout the interval where p, q, g are continuous.

Ex: $y''-y=0$, $y(0)=2$, $y'(0)=-1$
we found a sol. $y=\frac{1}{2}e^t+\frac{3}{2}e^{-t}$ it exists for $-\infty<t<\infty$ and is unique.

Ex $(t^2-3t)y''+ty'-(t+3)y=0$; $y(1)=2$, $y'(1)=1$
find the longest interval in which the sol. is certain to exist.

Sol: $y''+\underbrace{\frac{1}{t-3}}_{p(t)}y'-\underbrace{\frac{t+3}{t(t-3)}}_{q(t)}y=\underbrace{0}_{g(t)}$

coeffs are discontinuous at $t=0$, $t=3$
So, sol exists for $0<t<3$
- longest interval containing $t_0=1$, where p, q, g are continuous

Note: $y''+p(t)y'+q(t)y=0$, $y(t_0)=0$, $y'(t_0)=0$ $\to$ $y=0$ - unique solution in an interval $\alpha<t<\beta$ about t_0

THM (principle of superposition)

$y''+p(t)y'+q(t)y=0$ If y_1, y_2 are two solutions, then $\boxed{y=c_1y_1+c_2y_2}$ (*) also a solution for any c_1, c_2.

Indeed:
$$\begin{aligned} &y_1''+py_1'+qy_1=0 \quad |\cdot c_1 \\ +\ &y_2''+py_2'+qy_2=0 \quad |\cdot c_2 \\ \hline &(c_1y_1+c_2y_2)''+p(c_1y_1+c_2y_2)'+q(c_1y_1+c_2y_2)=0 \end{aligned}$$

So, starting with y_1, y_2, we construct an infinite family of sol. (*). Do we get all solutions?
- can we satisfy init. cond. $y(t_0)=y_0$, $y'(t_0)=y_0'$?

$$\begin{aligned} y(t_0)&=\\ y'(t_0)&\end{aligned}\boxed{\begin{aligned} c_1y_1(t_0)+c_2y_2(t_0)&=y_0 \\ c_1y_1'(t_0)+c_2y_2'(t_0)&=y_0' \end{aligned}}$$

- can be solved for any y_0, y_0' for c_1, c_2 iff $\underbrace{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}_{y_1(t_0)y_2'(t_0)-y_2(t_0)y_1'(t_0)=:W}\neq 0$

$W=W[y_1,y_2](t_0)$ - Wronskian determinant (or just Wronskian) of the solutions y_1, y_2 (at t_0)

So: if y_1, y_2 two sol. of $y''+p(t)y'+q(t)y=0$ then one can find c_1, c_2 such that $y = c_1 y_1(t) + c_2 y_2(t)$ satisfies the init. cond. $y(t_0)=y_0, y'(t_0)=y_0'$ for any y_0, y_0' iff the Wronskian $W[y_1,y_2] = y_1 y_2' - y_2 y_1'$ is nonzero at t_0.

Ex: $y''+5y'+6y=0$ $y_1 = e^{-2t}, y_2 = e^{-3t}$ are solutions.

Wronskian: $W[e^{-2t}, e^{-3t}] = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} = -e^{-5t}$ $\neq 0$ for all t.

$\Rightarrow$ y_1, y_2 can be used to construct sol. to the eq. with any init. cond. at any t!

THM Suppose y_1, y_2 are sol. of $y''+p(t)y'+q(t)y=0$

The $\underbrace{\text{family}}_{\text{2-parameter}}$ of solutions $y = c_1 y_1(t) + c_2 y_2(t)$ includes every sol. of the eq. iff $W[y_1,y_2](t_0) \neq 0$ for some t_0.

So, $W[y_1,y_2](t_0) \neq 0$ $\Leftrightarrow$ $y = c_1 y_1 + c_2 y_2$ is the General solution.

in this case, y_1 and y_2 are said to form a fundamental set of solutions.

Ex suppose $y_1(t) = e^{r_1 t}, y_2(t) = e^{r_2 t}$ - solutions of $y''+p(t)y'+q(t)y=0$

show that, if $r_1 \neq r_2$, y_1 and y_2 form a fund. set of sol.

Sol: $W = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = \underbrace{(r_2 - r_1)}_{\neq 0} e^{(r_1+r_2)t} \neq 0.$

Ex: show that $y_1 = t^{1/2}, y_2 = t^{-1}$ ~~form sol. of 2t~~ a fund. set of sol. for $2t^2y''+3ty'-y=0$, $t>0$

Sol: $y_1' = \frac{1}{2}t^{-1/2}, y_1'' = -\frac{1}{4}y^{-3/2}$ $2t^2y_1''+3ty_1'-y_1 = (-\frac{2}{4}+\frac{3}{2}-1)t^{1/2} = 0$

$y_2' = -t^{-2}, y_2'' = 2t^{-3}$ $2t^2y_2''+3ty_2'-y_2 = (4-3-1)t^{-1} = 0$

So, y_1, y_2 are, indeed, solutions

$W[y_1,y_2] = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}t^{-1/2} & -t^{-2} \end{vmatrix} = -\frac{3}{2}t^{-3/2} \neq 0$ for $t>0$ $\Rightarrow$ y_1, y_2 - FSS $\Rightarrow$ $y = c_1 t^{1/2} + c_2 t^{-1}$ - general sol.

- $y''+p(t)y'+q(t)y=0$

(*) let y_1 be the sol. satisfying $y(t_0)=1, y'(t_0)=0$

let y_2 — " — $y(t_0)=0, y'(t_0)=1$

Then y_1, y_2 form a FSS.

(because $W[y_1,y_2](t_0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$)

Ex: $y''-y=0$

$\underbrace{y_1 = e^t, y_2 = e^{-t}}_{\text{FSS (does not satisfy (*) at } t_0=0)}$

$\underbrace{\tilde{y}_1 = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \cosh t, \quad \tilde{y}_2 = \frac{1}{2}e^t - \frac{1}{2}e^{-t} = \sinh t}_{\text{another FSS, satisfying (*)}}$

• $y'' + p(t)y' + q(t)y = 0$ p, q continuous real ~~sth~~ fun. 04/20/2018

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If $y = u(t) + iv(t)$ - complex-valued solution, then u and v are also solutions.

Abel's THM

$y'' + p(t)y' + q(t)y = 0$; let p, q be continuous ~~for~~ $\alpha < t < \beta$ and y_1, y_2 be solutions.

then $W[y_1, y_2](t) = c \exp\left(-\int p(t)\,dt\right)$ (#). c - const., depends on y_1, y_2, but not on t.

$W[y_1, y_2](t)$ is either zero for all t (if $c = 0$), or else nonzero everywhere in $\alpha < t < \beta$.

Idea:

$$
\begin{array}{l}
y_1'' + p y_1' + q y_1 = 0 \mid \cdot (-y_2) \\
+ \quad y_2'' + p y_2' + q y_2 = 0 \mid \cdot y_1 \\
\hline
\underbrace{y_1 y_2'' - y_1'' y_2}_{(y_1 y_2' - y_2 y_1')'} + p(y_1 y_2' - y_2 y_1') = 0
\end{array}
$$

So: $W' + p(t) W = 0 \longrightarrow W = c e^{-\int p(t)\,dt}$

Ex: $2t^2 y'' + 3t y' - y = 0$, $t > 0$ $y_1 = t^{1/2}$, $y_2 = t^{-1}$. Verify that W is given by Abel's f-la (#).

Sol: we found $W = -\frac{3}{2} t^{-\frac{3}{2}}$

$\hookrightarrow y'' + \underbrace{\frac{3}{2} t^{-1}}_{p} y' \underbrace{- \frac{1}{2t^2}}_{q} y = 0 \quad \rightsquigarrow W = c e^{-\int \frac{3}{2} t^{-1} dt} = c e^{-\frac{3}{2} \ln t} = c \cdot t^{-3/2}$ ✓ $c = -\frac{3}{2}$