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3.4 Repeated roots. Reduction of order.

$ay'' + by' + cy = 0$ char. eq. $ar^2 + br + c = 0$

Consider the case $r_1 = r_2$ - happens when $b^2 - 4ac = 0$; then $r_1 = r_2 = -\frac{b}{2a}$

both roots yield $y_1 = e^{-\frac{b}{2a}t}$ how to find a second solution?

Ex: $y'' + 4y' + 4y = 0$ $\quad r^2 + 4r + 4 = 0$, $\underset{=}{}(r+2)^2$ $\quad r_1 = r_2 = -2$ $\quad y_1(t) = e^{-2t}$

Idea: look for solutions of form $y = \underbrace{v(t)}_{\text{undetermined function}} y_1(t)$ (*)

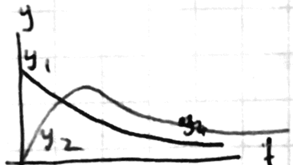
$y = ve^{-2t}$ (*) $\rightsquigarrow y' = v'e^{-2t} - 2ve^{-2t} \rightsquigarrow y'' = v''e^{-2t} - 2v'e^{-2t} - 2v'e^{-2t} + 4ve^{-2t}$

$y'' + 4y' + 4 = e^{-2t}((v'' - \cancel{4v'} + \cancel{4v}) + 4(\cancel{v'} - \cancel{2v}) + \cancel{4v}) = v''e^{2t} \overset{\text{WANT}}{=} 0$

So, (*) - solution iff $v'' = 0 \xrightarrow{\text{integrate}} v' = C_2 \rightarrow v = C_2 t + C_1$

$\rightarrow y =$ ~~$C_2 t e$~~ $\underbrace{C_2 t e^{-2t}}_{y_2} + \underbrace{C_1 e^{-2t}}_{y_1}$ $\quad y_2(t) = te^{-2t}$ - new solution

$W[y_1, y_2] = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{vmatrix} = e^{-4t}(1-2t) + 2te^{-4t} = e^{-4t} \neq 0$ So, $\left.\begin{matrix} y_1 = e^{-2t} \\ y_2 = te^{-2t} \end{matrix}\right\}$ FSS



$y = c_1 y_1 + c_2 y_2$ - general sol. All solutions $\xrightarrow[t\to\infty]{} 0$

Generally: assume we have a repeated root $r_1 = r_2 = -\frac{b}{2a}$ (and $b^2 - 4ac = 0$)

$y_1 = e^{-\frac{b}{2a}t}$ try to find solutions of form $y = v(t)\, y_1$:

$y = vy_1 \rightsquigarrow y' = v'y_1 + vy_1' \rightsquigarrow y'' = v''y_1 + 2v'y_1' + vy_1''$

$= e^{-\frac{b}{2a}}v \qquad = e^{-\frac{b}{2a}}(v' - \frac{b}{2a}v) \qquad = e^{-\frac{b}{2a}}(v'' - \frac{b}{a}v' + \frac{b^2}{4a^2}v)$

$ay'' + by' + cy = e^{-\frac{b}{2a}}(a(v'' - \cancel{\frac{b}{a}v'} + \frac{b^2}{4a^2}v) + b(\cancel{v'} - \frac{b}{2a}) + cv) = e^{-\frac{b}{2a}}(av'' + \underbrace{(-\frac{b^2}{4a} + c)}_{0}v) \overset{\text{WANT}}{=} 0$

So: $v''(t) = 0 \rightsquigarrow v'(t) = C_2 \rightsquigarrow v(t) = C_2 t + C_1$

$\rightsquigarrow y = \underbrace{C_1 e^{-\frac{b}{2a}t}}_{y_1} + C_2 \underbrace{te^{-\frac{b}{2a}t}}_{y_2}$ $\quad W[y_1, y_2] = \begin{vmatrix} e^{-\frac{b}{2a}t} & te^{-\frac{b}{2a}t} \\ -\frac{b}{2a}e^{-\frac{b}{2a}t} & (1 - \frac{b}{2a}t)e^{-\frac{b}{2a}t} \end{vmatrix} = e^{-\frac{b}{a}} \neq 0$

y_1, y_2 - FSS $\Rightarrow$ - general sol.

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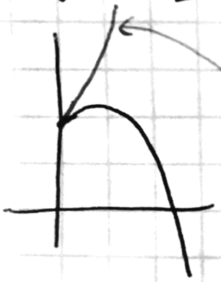
Ex: $y'' - y' + \frac{y}{4} = 0$, $y(0) = 2$, $y'(0) = \frac{1}{3}$ - solve the IVP

Sol: $r^2 - r + \frac{1}{4} = 0$ $= (r - \frac{1}{2})^2$ $\quad r_1 = r_2 = \frac{1}{2}$ $\quad \boxed{y = c_1 e^{\frac{t}{2}} + c_2 t e^{\frac{t}{2}}}$ gen. sol.

$$y' = \frac{1}{2} c_1 e^{\frac{t}{2}} + c_2 (1 + \frac{t}{2}) e^{\frac{t}{2}}$$

$$\begin{aligned} y(0) &= c_1 = 2 \\ y'(0) &= \frac{1}{2} c_1 + c_2 = \frac{1}{3} \end{aligned} \sim \begin{aligned} c_1 &= 2 \\ c_2 &= -\frac{2}{3} \end{aligned} \rightsquigarrow y = 2e^{t/2} - \frac{2}{3} t e^{t/2} \xrightarrow[t \to \infty]{} -\infty$$



If init. cond.: $\begin{aligned} y(0) &= 2 \\ y'(0) &= 2 \end{aligned} \sim \begin{aligned} c_1 &= 2 \\ c_2 &= 1 \end{aligned} \to y = 2e^{\frac{t}{2}} + t e^{\frac{t}{2}} \xrightarrow[t \to \infty]{} +\infty$

$r > 0 \to y \xrightarrow[t \to \infty]{} \pm\infty$

$r < 0 \to y \xrightarrow[t \to \infty]{} 0$

$r = 0 \to y$ is a linear function of t

Summary for $ay'' + by' + cy = 0$. Let r_1, r_2 be roots of $ar^2 + br + c = 0$

- if $r_1 \neq r_2$ real , $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ - gen sol.
- r_1, r_2 complex conjugates $\lambda \pm i\mu$, $y = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t$
- $r_1 = r_2$, $y = c_1 e^{rt} + c_2 t e^{rt}$

Reduction of order more generally.

$y'' + p(t) y' + q(t) y = 0$ suppose we know ~~one~~ sol. $y_1(t)$ (not everywhere zero)

try $y = v(t) y_1(t) \rightsquigarrow y' = v' y_1 + v y_1' \to y'' = v'' y_1 + 2v' y_1' + v y_1''$

$$y'' + py' + qy = y_1 v'' + (2y_1' + py_1) v' + \underbrace{(y_1'' + py_1' + qy_1)}_{0} v = 0$$

$\to \boxed{y_1 v'' + (2y_1' + py_1) v' = 0}$ (#) - 1st order ODE for v' ~ solve for v' $\xrightarrow{\text{integrate}} v \rightsquigarrow$ we get the sol. $y = vy_1$

Ex: $2t^2 y'' + 3ty' - y = 0$, $t > 0$, $y_1(t) = t^{-1}$ is a sol. find a FSS.

Sol: ~~$y = \ldots$~~ $\quad y'' + \underbrace{\frac{3}{2t}}_{p} y' \underbrace{- \frac{1}{2t^2}}_{q} y = 0 \qquad y = vt^{-1}$ a sol. iff

$$(\#): \; t^{-1} v'' + \underbrace{\left(-\frac{2}{t^2} + \frac{3}{2t^2}\right)}_{-\frac{1}{2t^2}} v' = 0$$

$v'' - \frac{1}{2t} v' = 0 \qquad$ set $v' = w$

$w' - \frac{1}{2t} w = 0 \rightsquigarrow \underset{v'}{w} = c_2 t^{1/2} \xrightarrow{\text{integrate}} v = \frac{2}{3} c_2 t^{3/2} + c_1$

$\frac{d}{dt} \ln|w| = +\frac{1}{2t}$

$$\to y = vt^{-1} = \frac{2}{3} c_2 \underbrace{t^{1/2}}_{y_2 \leftarrow \text{new sol.}} + c_1 \underbrace{t^{-1}}_{y_1}$$

$$W[y_1, y_2] = \begin{vmatrix} t^{-1} & t^{1/2} \\ -t^{-2} & \frac{1}{2} t^{-1/2} \end{vmatrix} = \frac{3}{2} t^{-3/2} \underset{t>0}{\neq} 0 \Rightarrow y_1, y_2 \text{ - FSS}$$