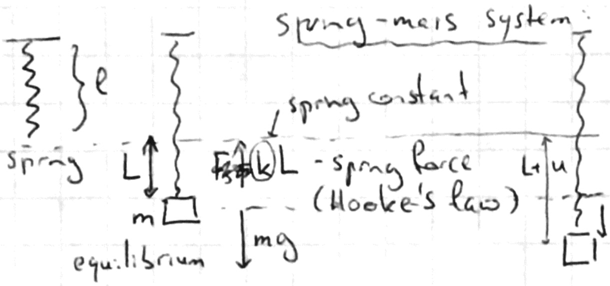
3.7 Mechanical and electrical vibrations

Spring-mass system:



$mg = kL$

$mu''(t) = f(t)$ ← net force (measured downward)

$= \underset{\text{weight}}{w} + \underset{\text{spring}}{F_s(t)} + \underset{\text{damping (resistance)}}{F_d(t)} + \underset{\text{applied external force}}{F(t)}$

$= mg - k(L+u(t)) - \underset{\text{damping constant}}{\gamma} u'(t) + F(t)$

(*)

$$\boxed{mu''(t) + \gamma u'(t) + ku(t) = F(t)}$$

$\underset{\text{init. displacement}}{u(0) = u_0}$, $\underset{\text{init. velocity}}{u'(0) = V_0}$

Ex 1: $\underset{w}{\overset{\text{mass weighing}}{4\,lb}}$ stretches a spring by $L = 2$ in. initial additional displacement $u_0 = 6$ in; then released ($u_0' = 0$)

viscuous resistance of the medium: $6\,lb/_{3\,ft/s}$. Formulate the initial problem.

Sol: in → ft: $L = \frac{2}{12} = \frac{1}{6}$ ft, $u_0 = \frac{6\text{ in}}{12} = \frac{1}{2}$ ft ; $g = 32\,ft/s^2$ $m = \frac{w}{g} = \frac{4\,lb}{32\,ft/s^2} = \frac{1}{8}\,\frac{lb \cdot s^2}{ft}$

$\gamma = \frac{6\,lb}{3\,ft/s} = 2\,\frac{lb \cdot s}{ft}$; $k = \frac{4\,lb}{\frac{1}{6}\,ft} = 24\,\frac{lb}{ft}$

So, eq(*): $\frac{1}{8}u''(t) + 2u'(t) + 24\,u(t) = \underbrace{0}_{\text{no external force}}$ or

$$\boxed{\begin{array}{l} u'' + 16u' + 192u = 0 \\ u(0) = \frac{1}{2} \quad u'(0) = 0 \leftarrow \text{init. cond.} \end{array}}$$

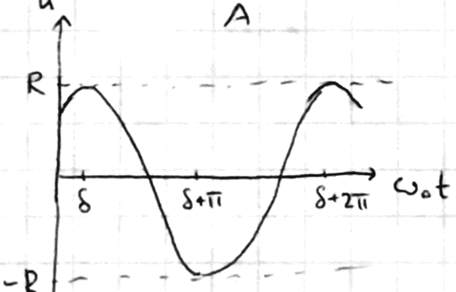
undamped free vibrations

$\underset{\text{no force applied}}{F(t) = 0}$, $\underset{\text{no damping}}{\gamma = 0}$ (*): $mu'' + ku = 0$ char. eq. $mr^2 + k = 0 \rightsquigarrow r = \pm i\sqrt{\frac{k}{m}}$

$u = A\cos\omega_0 t + B\sin\omega_0 t$, $\omega_0^2 = \frac{k}{m}$ A, B determined by init. cond.

$u = R\cos(\omega_0 t - \delta)$

$= \underbrace{R\cos\delta}_{A}\cos\omega_0 t + \underbrace{R\sin\delta}_{B}\sin\omega_0 t$, $R = \sqrt{A^2 + B^2}$, $\tan\delta = \frac{B}{A}$



← periodic ("simple harmonic") motion of the mass

period: $T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$

$\omega_0 = \sqrt{\frac{k}{m}}$ (in radians/unit time) - natural (circular) frequency of the vibration

R - amplitude

δ - phase

- amplitude does not decrease with time ~ no damping (dissipation of energy)
- ω_0, T do not depend on initial conditions! (R, δ do depend)
- as m increases, T increases. as k increases, T decreases.

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Ex^2 mass with $w = 10\,lb$ stretches the spring by $L = 2\,in$. $\approx \frac{1}{6} ft$
if it is stretched by further $u_0 = 2\,in$ $\approx \frac{1}{6} ft$, and set in motion with upward velocity $u_0' = -1\, ft/s$,
determine $u(t)$, T, R, δ

Sol: $k = \frac{w}{L} = \frac{10}{1/6} = 60 \frac{lb}{ft}$, $m = \frac{w}{g} = \frac{10}{32} \frac{lb \cdot s^2}{ft}$ $\qquad$ $(*)$: $\frac{10}{32} u'' + 60 u = 0$

$\rightarrow u'' + 192 u = 0$

$r^2 + 192 = 0 \qquad r = \pm i\, 8\sqrt{3}$

$u = A\cos(8\sqrt{3}t) + B\sin(8\sqrt{3}t)$

$u(0) = A = \frac{1}{6} ft$

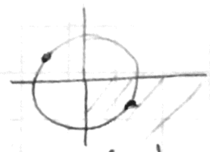
$u'(0) = 8\sqrt{3} B = -1 \, ft/s$

$\rightsquigarrow \boxed{u = \frac{1}{6}\cos(8\sqrt{3}t) - \frac{1}{8\sqrt{3}}\sin(8\sqrt{3}t)} \; (ft)$

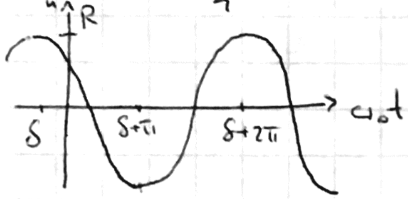
$\omega_0 = 8\sqrt{3} \approx 13.856 \frac{rad}{s} \qquad T = \frac{2\pi}{\omega_0} \approx 0.4535$

$R = \sqrt{\frac{1}{36} + \frac{1}{192}} = \sqrt{\frac{19}{576}} \approx 0.182 ft$; $\tan\delta = -\frac{6}{8\sqrt{3}} = -\frac{\sqrt{3}}{4}$

- two solutions: in II nd quadrant and in IVth. $\cos\delta > 0$, $\sin\delta < 0$ $\rightarrow$ IVth quadrant



$\delta = -\arctan\frac{\sqrt{3}}{4} \approx -0.40864\, rad$



- undamped free vibration

Damped free vibrations

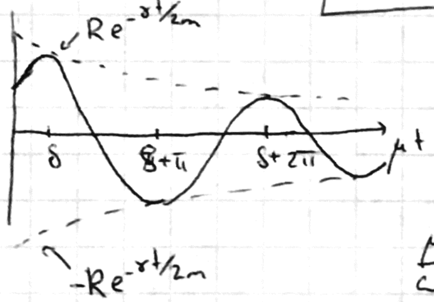
$mu'' + \gamma u' + ku = 0$ $\qquad$ char. eq. $mr^2 + \gamma r + k = 0 \rightsquigarrow r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4km}}{2m} = \frac{\gamma}{2m}\left(-1 \pm \sqrt{1 - \frac{4km}{\gamma^2}}\right)$

(1) $\gamma^2 - 4km > 0$, $u = Ae^{r_1 t} + Be^{r_2 t}$ $\qquad (r_1, r_2 < 0)$

(2) $\gamma^2 - 4km = 0$, $u = (A + Bt)e^{-\frac{\gamma}{2m}t}$

(3) $\gamma^2 - 4km < 0$, $u = e^{-\frac{\gamma t}{2m}}(A\cos\mu t + B\sin\mu t)$ $\qquad \boxed{\mu = \frac{1}{2m}\sqrt{4km - \gamma^2}} > 0$

$= \boxed{Re^{-\frac{\gamma t}{2m}}\cos(\mu t - \delta)}$ $\quad$ where $A = R\cos\delta$, $B = R\sin\delta$



- damped vibration

Cosine wave whose amplitude decreases exponentially

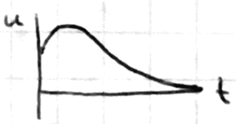
μ - "quasi-frequency"

$T_d = \frac{2\pi}{\mu}$ - "quasi-period".

$\frac{\mu}{\omega_0} = \left(1 - \frac{\gamma^2}{4km}\right)^{1/2} \approx 1 - \frac{\gamma^2}{8km}$ (for γ small)

$\frac{T_d}{T} = \frac{\omega_0}{\mu} = \left(1 - \frac{\gamma^2}{4km}\right)^{-1/2} \approx 1 + \frac{\gamma^2}{8km}$ $\qquad$ small damping increases the quasi-period

For $\gamma = 2\sqrt{km}$ - critically damped system, $T_d \rightarrow \infty$, $\mu \rightarrow 0$



$\gamma > 2\sqrt{km}$ - overdamped system