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Canonical quantization

Promote φ_n, π_n to operators $\hat\varphi_n, \hat\pi_n$ s.t. $[\hat\pi_n, \hat\varphi_m] = -i\delta_{n,-m}$ (we set $\hbar = 1$)

introduce creation/annihilation operators a_n, ~~[illegible]~~ $\bar a_n$, $n \neq 0$:

$(\hat a_n)^\dagger = \hat a_{-n}$
$(\hat{\bar a}_n)^\dagger = \hat{\bar a}_{-n}$

$$\left.\begin{array}{l} \hat\varphi_n = \frac{i}{n}(-\hat a_{-n} + \hat{\bar a}_n) \\ \hat\pi_n = \frac{\hat a_{-n} + \hat{\bar a}_n}{2} \end{array}\right\} \text{ with }$$

~~$[\hat a_n, \hat a_m] = 0 = [\hat{\bar a}_n, \hat{\bar a}_m]$~~

$$\left.\begin{array}{l} [\hat a_n, \hat a_m] = n\delta_{n,-m} \\ [\hat{\bar a}_n, \hat{\bar a}_m] = n\delta_{n,-m} \\ [\hat a_n, \hat{\bar a}_m] = 0 \end{array}\right\} (*) \quad , \quad \hat H = \sum_{n\neq 0} \frac{\hat a_{-n}\hat a_n + \hat{\bar a}_{-n}\hat{\bar a}_n}{2} + (\hat\pi_0)^2$$

may define $\hat a_0 = \hat{\bar a}_0 := \hat\pi_0$

then $\hat H = \frac{1}{2}\sum_{n\in\mathbb{Z}}(\hat a_{-n}\hat a_n + \hat{\bar a}_{-n}\hat{\bar a}_n)$, total momentum operator $\hat P = \frac{1}{2}\sum_{n\in\mathbb{Z}}(\hat a_{-n}\hat a_n - \hat{\bar a}_{-n}\hat{\bar a}_n)$

- Lie algebra $\underbrace{\mathrm{Span}_{\mathbb{C}}(\{\hat a_n\}_{n\in\mathbb{Z}}, \mathbb{K})}_{\text{"Heisenberg Lie algebra"}}$ with comm. rel. $[\hat a_n, \hat a_m] = n\delta_{n,-m}$
  $[\hat a_n, \mathbb{K}] = 0$

$\cong$ central extension of the abelian Lie alg. of formal Laurent series $\{f(z) = \sum_{n\in\mathbb{Z}} f_n \underbrace{z^{-n}}_{\text{"}a_n\text{"}}\}$

with $[f,g]' = \mathbb{K}\cdot \underbrace{\mathrm{res}_{z=0}}_{\text{coeff. of } z^{-1}dz}(f\,dg)$

we have $[\hat H, \hat a_n] = -n\hat a_n$ $\quad [\hat P, \hat a_n] = -n\hat a_n$
$[\hat H, \hat{\bar a}_n] = -n\hat{\bar a}_n$ $\quad [\hat P, \hat{\bar a}_n] = +n\hat{\bar a}_n$

for $n>0$:	annihilation operator	creation operator
right mover	$\hat a_n$	$\hat a_{-n}$
left mover	$\hat{\bar a}_n$	$\hat{\bar a}_{-n}$

Space of states: $\mathcal{H} = \mathcal{H}_{\text{free particle}} \otimes \bigotimes_{n\neq 0} \mathcal{H}_{\text{Harm. osc.}, \omega_n = |n|}$ $= \mathrm{Span}_{\mathbb{C}}\{\prod_{n,\bar n>0}(\hat a_{-n})^{k_n}(\hat{\bar a}_{-\bar n})^{k_{\bar n}}$

$$= \mathrm{Span}\left\{ \underbrace{\prod_{i=1}^{r}\hat a_{-n_i}\prod_{j=1}^{s}\hat{\bar a}_{-\bar n_j}|\pi_0\rangle}_{|\pi_0; \{n_i\}, \{\bar n_j\}\rangle} \;\middle|\; \begin{array}{l} 1\le n_1\le n_2\le \dots \le n_r \\ 1\le \bar n_1\le \bar n_2\le\dots\le \bar n_s \\ \pi_0\in\mathbb{R}\end{array}\right\}$$

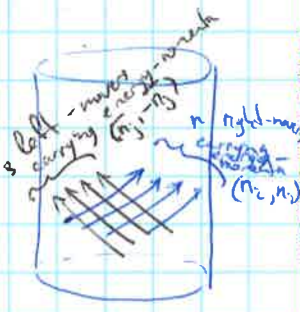
← "(r+s)-particle state"

occupation numbers

$$= \mathrm{Span}\{\prod_{n\geq 1}(\hat a_{-n})^{k_n}\cdot\prod_{\bar n\geq 1}(\hat{\bar a}_{-\bar n})^{k_{\bar n}}\ |\pi_0\rangle\}$$

- normally-ordered operators $:\hat H:$, $:\hat P:$ – put annihilation operators $a_{>0}, \bar a_{>0}$ to the right, creation operators $a_{<0}, \bar a_{<0}$ to the left.
  $:\cdots: \circlearrowleft \text{Free Assoc Alg}(\{a_n, \bar a_n\}_{n\in\mathbb{Z}})$
  $\mathcal{O} \longmapsto :\mathcal{O}:$

We have: $:\hat H: |\pi_0; \{n_i\}, \{\bar n_j\}\rangle = (\pi_0^2 + \sum_i n_i + \sum_j \bar n_j)\,|\pi_0; \{n_i\}, \{n_j\}\rangle$
$:P: |\pi_0; \{n_i\}, \{\bar n_j\}\rangle = (\sum_i n_i - \sum_j \bar n_j)\ |\pi_0; \{n_i\}, \{n_j\}\rangle$



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Aside: Schrödinger vs Heisenberg picture in quantum mechanics

Schrödinger: $|\psi\rangle_{(t)} = \underbrace{e^{-\frac{i}{\hbar}\hat H(t-t_0)}}_{U(t-t_0)}|\psi\rangle_{t_0}$

evolution acts on states

t_0 - reference time

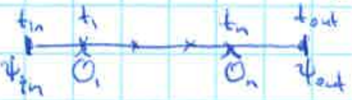
Heisenberg: evol. acts on observables

$$\hat O_t = U^{-1}(t-t_0)\,\hat O_{t_0}\,U(t-t_0)$$

← Heisenberg (time-dependent) observable

$$-i\hbar\frac{d}{dt}\hat O_t = [\hat H, \hat O_t]$$

$\langle\psi_{out}|$ Correlator:



$$= \langle\psi_{out}|U(t_{out}-t_n)\overset{U(t_0-t_n)\hat O_n U(t_n-t_0)}{\hat O_n}U(t_n-t_{n-1})\cdots\hat O_1\,U(t_1-t_{in})|\psi_{in}\rangle$$

$$= \underbrace{\langle\widetilde{\psi_{out}}|}_{\langle\psi_{out}|U(t_{out}-t_0)}\ \hat O_n(t_n)\ \underbrace{\hat O_{n-1}(t_{n-1})}_{U(t_0-t_{n-1})\hat O_{n-1}U(t_{n-1}-t_0)}\cdots\hat O_1(t_1)\ \underbrace{|\widetilde{\psi_{in}}\rangle}_{U(t_0-t_{in})|\psi_{in}\rangle}$$

time-dependent observables (Heisenberg)

it is a time-ordered product of operators, $t_n > t_{n-1} > \dots > t_1$

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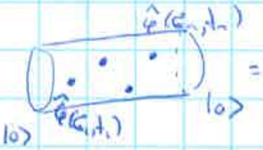
Back to the free boson on the cylinder

Time-dependent field operators:

$$\underbrace{\hat\varphi(\sigma)}_{\sum_n \hat\varphi_n e^{in\sigma}} \longrightarrow \hat\varphi(\sigma,t) = \underbrace{e^{i\hat H t}}_{U^{-1}(t)}\hat\varphi(\sigma)\underbrace{e^{-i\hat H t}}_{U(t)} = \hat\varphi_0 + 2t\hat\pi_0 + \sum_{n\neq 0}\frac{i}{n}\left(-\hat a_{-n}e^{in(\sigma+t)} + \hat{\bar a}_n e^{in(\sigma-t)}\right)$$

$\hat\varphi_n \uparrow \frac{i}{n}(-\hat a_{-n} + \hat{\bar a}_n)$

Can consider correlators



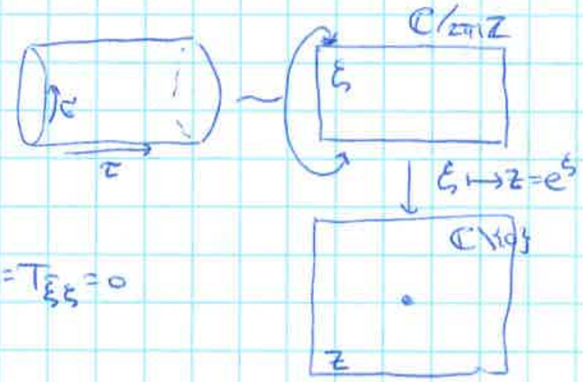
$= \langle 0|\hat\varphi(\sigma_n,t_n)\cdots\hat\varphi(\sigma_1,t_1)|0\rangle$
$t_n > \dots > t_1$

Euclidean cylinder

$$S_{Eucl} = \frac{\varkappa}{2}\int d\tau\, d\sigma\, \left((\partial_\tau\varphi)^2 + (\partial_\sigma\varphi)^2\right)$$

$$\longrightarrow\ = 2\varkappa\int_{\mathbb{C}/2\pi i\mathbb{Z}}\frac{i}{2}\,d\xi\, d\bar\xi\ \partial_\xi\varphi\ \partial_{\bar\xi}\varphi$$

complex coord $\xi = \tau + i\sigma$
$\bar\xi = \tau - i\sigma$



Stress-energy tensor $T_{\xi\xi} = \varkappa(\partial_\xi\varphi)^2$, $T_{\bar\xi\bar\xi} = \varkappa(\partial_{\bar\xi}\varphi)^2$, $T_{\xi\bar\xi} = T_{\bar\xi\xi} = 0$

Minkowski metric → Euclidean metric

Formal substitution $t \longmapsto -i\tau$ ("Wick rotation")

evolution $e^{-i\hat H t} \longmapsto e^{-\hat H\tau}$

Space of states $\mathcal{H}$ and Hamiltonian $\hat H$ do not change!

(Heisenberg) field operator: $\hat\varphi(\xi,\bar\xi) = \hat\varphi_0 - i\hat\pi_0(\xi+\bar\xi) + i\sum_{n\neq 0}\frac{\hat a_n e^{-n\xi} + \hat{\bar a}_n e^{-n\bar\xi}}{n}$

$$= \boxed{\hat\varphi_0 - i\hat\pi_0\log(z\bar z) + \sum_{n\neq 0}\frac{i}{n}\left(\hat a_n z^{-n} + \hat{\bar a}_n\bar z^{-n}\right)}$$

Aside Wick's lemma <in the language of operators>

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Let $\mathcal{A} = \mathrm{Span}_{\mathbb{C}}(\{a_k, a_k^+\}, \mathbb{K})$, $[a_i, a_j^+] = \mathbb{K}\,\delta_{ij}$; let $\{A_p \in \mathcal{A}\}$ be a collection of elements (~~[illegible]~~ lin. comb. of creation/annihil. operators)

- Heisenberg Lie algebra

Let $\{g_{pq} \in \mathbb{C}\}$ be ~~[illegible]~~ defined by $A_p A_q - :A_p A_q: = \underbrace{g_{pq}}_{\text{"propagator"}}\mathbb{K}$ in $U(\mathcal{A})$

Then:

$$\boxed{A_{p_1}A_{p_2}\cdots A_{p_n} = \sum_{\substack{\{\alpha_1,\beta_1\}\sqcup\cdots\sqcup\{\alpha_s,\beta_s\}\\ \subset\{1,\dots,n\}}} \mathbb{K}^s\, g_{p_{\alpha_1}p_{\beta_1}}\cdots g_{p_{\alpha_n}p_{\beta_n}} :\prod_{\substack{i=1\\ i\notin \cup\{\alpha_k,\beta_k\}}}^{n} A_{p_i}:}$$

- a "matching"
- collection of 2-element subsets

Ex: $A_a A_b = \mathbb{K} g_{ab} + :A_a A_b:$

$$A_a A_b A_c = ~~\mathbb{K} g_{ab} + \mathbb{K} g_{ac}~~ + \mathbb{K} g_{ab} \underbrace{A_c}_{=:A_c:} + \mathbb{K} g_{ac} A_b + \mathbb{K} g_{bc} A_a + :A_a A_b A_c:$$

$:A_a A_b A_c A_d:$

$$A_a A_b A_c A_d = \mathbb{K}^2 g_{ab} g_{bc} + \mathbb{K}^2 g_{ac} g_{bd} + \mathbb{K}^2 g_{ad} g_{bc} + \mathbb{K} g_{ab} :A_c A_d: \text{ (+ 5 similar terms)} + :A_a A_b A_c A_d:$$

$A_a A_b A_c A_d$

---

back to free boson on Euclidean cylinder (or $\mathbb{C}\setminus\{0\}$)

Recall: $\hat\varphi(z,\bar z) = \varphi_0 - i\pi_0 \log(z\bar z) + \sum_{n\neq 0} \frac{i}{n}\left(\hat a_n z^{-n} + \hat{\bar a}_n \bar z^{-n}\right)$

Propagator for $|z| > |w|$,

$$\hat\varphi(z,\bar z)\,\hat\varphi(w,\bar w) - :\hat\varphi(z,\bar z)\,\hat\varphi(w,\bar w): = \sum_{n>0} \frac{i}{n^2}\Big(\underbrace{[\hat a_n, \hat a_{-n}]}_{n} z^{-n} w^n + \underbrace{[\hat{\bar a}_n, \hat{\bar a}_{-n}]}_{n} \bar z^{-n} \bar w^n\Big) - i\underbrace{[\hat\pi_0, \hat\varphi_0]}_{-i} \log(z\bar z)$$

$$= \sum_{n>0} \frac{1}{n}\left(\left(\frac{w}{z}\right)^n + \left(\frac{\bar w}{\bar z}\right)^n\right) - \log(z\bar z) = -\log\left(1-\frac{w}{z}\right) - \log\left(1-\frac{\bar w}{\bar z}\right) - \log(z\bar z) = \boxed{-2\log|z-w|} = g(\{z,\bar z\};\{w,\bar w\})$$

- we supplement the normal ordering prescription by putting $\hat\varphi_0$ to the left of $\hat\pi_0$.

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Radial ordering of products of local operators

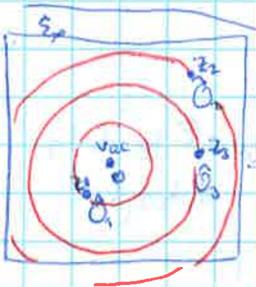
$$R~~\hat\varphi(z,\bar z)\,\hat\varphi~~ \; R\left(\hat O_1(z_1,\bar z_1)\cdots \hat O_n(z_n,\bar z_n)\right) := \hat O_{\sigma_1}(z_{\sigma_1},\bar z_{\sigma_1})\cdots \hat O_{\sigma_n}(z_{\sigma_n},\bar z_{\sigma_n})$$

where $\sigma \in S_n$ s.t. $|z_{\sigma_1}| > \cdots > |z_{\sigma_n}|$

where $\hat O_k(z_k,\bar z_k) =$ (1) $\hat\varphi(z_k,\bar z_k)$ ~~any derivative~~

(2) any derivative of $\hat\varphi(z_k,\bar z_k)$

(3) normally-ordered product of derivatives of $\hat\varphi(z_k,\bar z_k)$ <or sum of those>



$\rightarrow \langle \mathrm{vac}|\hat O_2(z_2)\hat O_3(z_3)\hat O_1(z_1)|\mathrm{vac}\rangle$

thus:

$$R\left(\hat\varphi(z,\bar z)\,\hat\varphi(w,\bar w)\right) = :\hat\varphi(z,\bar z)\,\hat\varphi(w,\bar w): - 2\log|z-w|\,\mathbb{1}$$

* 2-point correlator

$$\langle \mathrm{vac}| R\,\hat\varphi(z,\bar z)\,\hat\varphi(w,\bar w)\underbrace{|\mathrm{vac}\rangle}_{|\pi_0=0\rangle} = -2\log|z-w| + \underbrace{C}$$

"infinite constant", $C = \langle \mathrm{vac}|\hat\varphi_0^2|\mathrm{vac}\rangle$ ill-defined!

- because $\hat\varphi_0$ is an unbounded (differential) operator $\hat\varphi_0 = i\frac{d}{d\pi_0}$ and $|\mathrm{vac}\rangle$ is not in the domain

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* correlators of field φ are ill-defined, due to the presence of 0-mode $\hat{\varphi}_0$ but correlators of fields $\partial\varphi$, $\bar\partial\varphi$ are well-defined!

$$i\partial\hat\varphi(z)=\sum_{n\in\mathbb{Z}}\hat a_n z^{-n-1}\ ,\quad i\bar\partial\hat\varphi(\bar z)=\sum_{n\in\mathbb{Z}}\hat{\bar a}_n \bar z^{-n-1}$$

$$\langle\partial\varphi(z)\partial\varphi(w)\rangle = \langle vac|R(\partial\hat\varphi(z)\partial\hat\varphi(w))|vac\rangle = \langle vac|\overbrace{-\frac{1}{(z-w)^2}}^{\partial_z\partial_w(-2\log|z-w|)}\mathbb{1}+\underbrace{:\partial\hat\varphi(z)\partial\hat\varphi(w):}_{(*)}|vac\rangle$$

$$\xrightarrow{}\ \sum a_{\geq 0}a_{\geq 0}+\sum a_{\leq 0}a_{\leq 0}+\sum a_{\leq 0}a_{\geq 0}$$

kills |vac⟩

kills ⟨vac|

kills |vac⟩

$$=\boxed{-\frac{1}{(z-w)^2}}$$

Exercise: check that $\langle vac|\partial\hat\varphi(z)\partial\hat\varphi(w)|vac\rangle$ diverges for $|z|<|w|$

also: $\langle\bar\partial\varphi(\bar z)\bar\partial\varphi(\bar w)\rangle=-\frac{1}{(\bar z-\bar w)^2}$,

$$\langle\partial\varphi(z)\,\bar\partial\varphi(\bar w)\rangle=0$$

n-point correlators - calculated by Wick's lemma: ~~Eg~~

Ex $\langle\partial\varphi(z_1)\partial\varphi(z_2)\partial\varphi(z_3)\partial\varphi(z_4)\rangle = \langle vac|R(-----)|vac\rangle = \langle vac|\overbrace{\partial\varphi(z_1)\partial\varphi(z_2)}^{-\frac{1}{(z_1-z_2)^2}}\cdot\overbrace{\partial\varphi(z_3)\partial\varphi(z_4)}+2$ similar terms

$+\overbrace{\partial\varphi(z_1)\partial\varphi(z_2)}:\partial\varphi(z_3)\partial\varphi(z_4):+5$ similar

$+:\partial\varphi(z_1)\cdots\partial\varphi(z_4):\ |vac\rangle$

only terms where all 4 operators are matched contribute

$$=\boxed{\frac{1}{z_{12}^2z_{34}^2}+\frac{1}{z_{13}^2z_{24}^2}+\frac{1}{z_{14}^2z_{23}^2}}\ ,\ \text{where } z_{ij}=z_i-z_j$$

Ex: $\langle\underbrace{\partial\varphi(z_1)\partial\varphi(z_2)}\underbrace{\bar\partial\varphi(\bar z_3)\bar\partial\varphi(\bar z_4)}\rangle=\boxed{\frac{1}{z_{12}^2\bar z_{34}^2}}$

Ex: $\langle\partial\varphi(z_1)\cdots\partial\varphi(z_6)\rangle=$

$$=\underbrace{\frac{-1}{z_{12}^2z_{34}^2z_{56}^2}+14\text{ similar terms}}_{5!!=5\cdot3\cdot1\text{ perfect matchings on 6 elements}}$$

Ex: $\langle\partial\bar\partial\varphi(z)\,\partial\varphi(w)\rangle$ $z\neq w$ $=\frac{\partial}{\partial\bar z}\underbrace{\langle\partial\varphi(z)\partial\varphi(w)\rangle}_{-\frac{1}{(z-w)^2}}\ \boxed{=0}$

Ex: ~~$\langle:\partial\varphi(z_1)\partial\varphi(z_2):\ \partial\varphi(z_3)\ \partial\varphi(z_4)\rangle=$~~

$$\langle\partial\varphi(z_1)\ :\partial\varphi(z_2)\partial\varphi(z_3):\ \partial\varphi(z_4)\rangle\overset{(*)}{\underset{?}{=}}\langle vac|\overbrace{\partial\varphi(z_1):\partial\varphi(z_2)}\overbrace{\partial\varphi(z_3):\partial\varphi(z_4)}\ \text{~~|vac⟩~~}+\partial\varphi(z_1):\partial\varphi(z_2)\partial\varphi(z_3):\partial\varphi(z_4)|vac\rangle$$

assume $|z_1|>|z_2|>|z_3|>|z_4|$

(no $:\overbrace{\partial\varphi(z_2)\partial\varphi(z_3)}:$ term!)

$$=\boxed{\frac{1}{z_{12}^2z_{34}^2}+\frac{1}{z_{13}^2z_{24}^2}}$$

← regular (real-analytic) as $z_2\to z_3$!

In particular, one can introduce a field $:\partial\hat\varphi(z)\partial\hat\varphi(z):=\lim\limits_{w\to z}:\partial\hat\varphi(z)\partial\hat\varphi(w):$

$$=\lim_{w\to z}\left(\partial\hat\varphi(z)\partial\hat\varphi(w)+\frac{\mathbb{1}}{(w-z)^2}\right)$$

it has well-defined correlators, e.g.

$(*)\rightsquigarrow\ \langle\partial\varphi(z_1)\ :\partial\varphi(z_2)\partial\varphi(z_2):\ \partial\varphi(z_4)\rangle=\frac{2}{z_{12}^2z_{24}^2}$

(Quantum) Stress-energy tensor:

$$\boxed{\begin{aligned}\hat T(z)&=-\frac{1}{2}:\partial\hat\varphi(z)\partial\hat\varphi(z):\\ \bar T(\bar z)&=-\frac{1}{2}:\bar\partial\hat\varphi(\bar z)\bar\partial\hat\varphi(\bar z):\end{aligned}}$$

(normal ordering removes an "infinite constant" from the definition of $\hat T$)

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Operator product expansions

$$R\,\partial\hat\varphi(z)\,\partial\hat\varphi(w) = -\frac{1}{(z-w)^2}\overset{\text{id} \in \text{End}\,\mathcal{H}}{\mathbb{1}} + \underbrace{:\partial\hat\varphi(z)\,\partial\hat\varphi(w):}_{\text{regular as } z\to w}$$

$$= \sum_{n\geq 0}\frac{1}{n!}(z-w)^n :\partial^{n+1}\varphi(w)\,\partial\varphi(w):$$

Similarly: $R\,\bar\partial\hat\varphi(\bar z)\,\bar\partial\hat\varphi(\bar w) = -\frac{1}{(\bar z-\bar w)^2}\mathbb{1} + \text{reg.}$, $R\,\partial\hat\varphi(z)\,\bar\partial\hat\varphi(\bar w) = \text{reg.}$

$$R\,\underbrace{\hat T(z)}_{:-\frac12\partial\hat\varphi(z)\partial\hat\varphi(z):}\,\partial\hat\varphi(w) = \frac{\partial\hat\varphi(z)}{(z-w)^2} + \underbrace{:-\frac12\partial\hat\varphi(z)\,\partial\hat\varphi(z)\,\partial\hat\varphi(w):}_{\text{reg}} \underset{\substack{\uparrow\\ \text{expand 1st term}\\ \text{at } w}}{=} \frac{\partial\hat\varphi(w)}{(z-w)^2} + \frac{\partial^2\hat\varphi(w)}{z-w} + \text{reg.}$$

$R\,\hat{\bar T}(\bar z)\,\bar\partial\hat\varphi(\bar w) = \cancel{\frac{\bar\partial\hat\varphi(\bar w)}{(\bar z-\bar w)^2}}$ complex conjugate , $R\,\hat T(z)\,\bar\partial\hat\varphi(\bar w) = \text{reg.} = R\,\hat{\bar T}(\bar z)\,\partial\hat\varphi(w)$

$$R\,\hat T(z)\,\hat T(w) = :-\frac12\partial\hat\varphi(z)\,\partial\hat\varphi(z):\;:-\frac12\partial\hat\varphi(w)\,\partial\hat\varphi(w): \overset{\text{Wick}}{=} \frac{\frac12\mathbb{1}}{(z-w)^4} + \frac{:\partial\hat\varphi(z)\,\partial\hat\varphi(w):}{(z-w)^2} + \text{reg.}$$

4 partial contractions
(one $\partial\hat\varphi$ from first : :, one $\partial\hat\varphi$ from second : :)

$$\underset{\substack{\nearrow\\ \text{expand}\\ \text{at } w}}{=} \boxed{\frac{\frac12\mathbb{1}}{(z-w)^4} + \frac{2\hat T(w)}{(z-w)^2} + \frac{\partial\hat T(w)}{z-w} + \text{reg.}}$$

$R\,\hat{\bar T}(\bar z)\,\hat{\bar T}(\bar w) =$ cx conjugate

$R\,\hat T(z)\,\hat{\bar T}(\bar w) = \text{reg.}$

Virasoro algebra (and its action on $\mathcal{H}$)

* Virasoro $\overset{\text{Lie}}{}$algebra: $\text{Span}_{\mathbb{C}}\{\hat L_n\}_{n\in\mathbb{Z}}$, $[\hat L_n, \hat L_m] := (n-m)\,\hat L_{n-m} + \overset{\text{"central charge"}}{c}\,\delta_{n,-m}\frac{n^3-n}{12}\hat{\mathbb{K}}$, $\hat{\mathbb{K}} = c\cdot\hat{\mathbb{1}}$

– the unique central extension of Witt Lie algebra of $\substack{\text{merom. v.f.}\\ \text{on } \mathbb{C}\setminus\{0\}}$

$$[f(z)\frac{\partial}{\partial z}, g(z)\frac{\partial}{\partial z}]^{\text{new}} = (fg' - gf')\frac{\partial}{\partial z} + \frac{1}{12}\hat{\mathbb{K}}\underset{\circlearrowleft}{\oint}\frac{dz}{2\pi i}f'''(z)\,g(z)$$

In a general CFT (not just free boson):

$$\text{conf. v.f.}\quad \varepsilon(z)\frac{\partial}{\partial z} + \bar\varepsilon(\bar z)\frac{\partial}{\partial\bar z} \rightsquigarrow \underset{\substack{\nearrow\\ \text{representation of v.f.}\\ \text{on states}}}{\rho}(\varepsilon\partial + \bar\varepsilon\bar\partial) := -\frac{1}{2\pi i}\underset{\circlearrowleft}{\oint}dz\;\varepsilon(z)\hat T(z) + \frac{1}{2\pi i}\underset{\circlearrowright}{\oint}d\bar z\;\bar\varepsilon(\bar z)\hat{\bar T}(\bar z) \quad \in \text{End}(\mathcal{H})$$

In particular, $\hat L_n := \rho(-z^{n+1}\frac{\partial}{\partial z}) = \frac{1}{2\pi i}\underset{\circlearrowleft}{\oint}dz\,z^{n+1}\,\hat T(z) \quad \in \text{End}(\mathcal{H})$

similarly, $\hat{\bar L}_n := \ldots$

Inverse formula - Fourier expansion for $\hat T$: $\hat T(z) = \sum_{n\in\mathbb{Z}}\hat L_n z^{-n-2}$, $\hat{\bar T}(\bar z) = \sum_{n\in\mathbb{Z}}\hat{\bar L}_n \bar z^{-n-2}$

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· Assume $$\boxed{R\,\hat{T}(z)\,\hat{T}(w) = \frac{c/2\ \hat{\mathbb{1}}}{(z-w)^4} + \frac{2\hat{T}(w)}{(z-w)^2} + \frac{\partial\hat{T}(w)}{z-w} + \text{reg.}}\ (*)$$

for some $c \in \mathbb{R}$

← general form of TT OPE in CFT

⟨We know $(*)$ holds for free boson, with $c=1$⟩

Then $(*)$ implies Virasoro relations for $\hat{L}_n$! Indeed,

$$[\hat{L}_n, \hat{L}_m]\psi = \cancel{\oint \frac{dz}{2\pi i} z^{n+1}} \oint \frac{dz}{2\pi i}\oint\frac{dw}{2\pi i} z^{n+1} w^{m+1}\,\hat{T}(z)\,\hat{T}(w) - \oint\oint\frac{dw}{2\pi i}\oint\frac{dz}{2\pi i} w^{m+1} z^{n+1}\,\hat{T}(w)\,\hat{T}(z) =$$

~~state at the origin~~

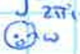
does not contribute

$$= \oint\frac{dw}{2\pi i}\oint\frac{dz}{2\pi i} z^{n+1} w^{m+} R\hat{T}(z)\hat{T}(w) = \oint\frac{dw}{2\pi i}\int\frac{dz}{2\pi i} z^{n+1} w^{m+1}\left(\frac{c/2\ \hat{\mathbb{1}}}{(z-w)^4} + \frac{2\hat{T}(w)}{(z-w)^2} + \frac{\partial\hat{T}(w)}{z-w}\right) + \text{reg.})$$

re-expand, $z = w + \alpha$

$$= \oint\frac{dw}{2\pi i}\oint\frac{d\alpha}{2\pi i}\underbrace{\left(w^{n+1} + (n+1)w^n\alpha + \frac{(n+1)n}{2} w^{n-1}\alpha^2 + \frac{(n+1)n(n-1)}{6} w^{n-2}\alpha^3 + \dots\right) w^{m+1}\left(\frac{c/2\ \hat{\mathbb{1}}}{\alpha^4} + \frac{2\hat{T}(w)}{\alpha^2} + \frac{\partial\hat{T}(w)}{\alpha} + \text{reg}\right)}_{\oint \text{ ~~need~~ evaluates the coeff. of } \alpha^{-1}}$$

$$= \oint\frac{dw}{2\pi i}\Big(\underbrace{w^{n+m+2}\partial\hat{T}(w)}_{\text{by parts}} + 2(n+1)\,w^{n+m+1}\,\hat{T}(w) + \frac{(n+1)n(n-1)}{12} c\, w^{n+m-1}\hat{\mathbb{1}}\Big) \boxed{= \big(\underbrace{(-n-m-2)+2n+2}_{n-m}\big)\hat{L}_{n+m} + \frac{n^3-n}{12} c\,\delta_{n,-m}\hat{\mathbb{1}}}$$



One says that field $\hat{\Phi}(z,\bar{z})$ is primary, of conformal weight $(h,\bar{h}) \in \mathbb{R}\times\mathbb{R}$ if it satisfies the OPEs

$$\boxed{R\,\hat{T}(z)\,\hat{\Phi}(w,\bar{w}) = \frac{h\,\hat{\Phi}(w,\bar{w})}{(z-w)^2} + \frac{\partial\hat{\Phi}(w,\bar{w})}{z-w} + \text{reg.}}, \quad \boxed{R\,\hat{\bar{T}}(\bar{z})\,\hat{\Phi}(w,\bar{w}) = \frac{\bar{h}\,\hat{\Phi}(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\hat{\Phi}(w,\bar{w})}{\bar{z}-\bar{w}} + \text{reg.}}$$

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"baby version of" Sugawara construction

Back to free boson: $\hat{T}(z) = -\frac{1}{2}:\partial\varphi(z)\partial\varphi(z):$ $\Rightarrow$ $$\boxed{\hat{L}_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} :\hat{a}_k\hat{a}_{n-k}:}\ (**), \quad \hat{\bar{L}}_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} :\hat{\bar{a}}_k\hat{\bar{a}}_{n-k}:$$

⟨note: normal ordering only affects $L_0, \bar{L}_0$⟩

I.e. we have an inclusion $\text{Vir}_{c=1} \hookrightarrow U^{(2)}\text{Heis}$

Note: $\hat{L}_0 + \hat{\bar{L}}_0 = \hat{H}$ - quantum Hamiltonian
$\hat{L}_0 - \hat{\bar{L}}_0 = \hat{P}$ - momentum operator

Exercise: ~~check~~ deduce Virasoro relations for $(**)$ from $[\hat{a}_n, \hat{a}_m] = n\,\delta_{n,-m}$

Note: $\hat{L}_{-2}|\text{vac}\rangle = \lim_{z\to 0}\hat{T}(z)|\text{vac}\rangle$
- always true in CFT

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Field - state correspondence

field $\hat{\Phi}(z,\bar{z}) \longmapsto$ state $\hat{\Phi}(0,0)|vac\rangle$

Ex: $i\partial\hat{\varphi} \longmapsto \lim_{z\to 0} i\partial\hat{\varphi}(z)|vac\rangle = \lim_{z\to 0} \underbrace{\sum_{n\in\mathbb{Z}} \hat{a}_n z^{-n-1}}_{\text{only term } n=-1 \text{ survives}} |vac\rangle = \hat{a}_{-1}|vac\rangle$

only term $n=-1$ survives $(a_{\geq 0}|vac\rangle = 0$ and for $n<-2$, $z^{-n-1}\to 0)$

$i\partial^2\hat{\varphi} \longmapsto \hat{a}_{-2}|vac\rangle$

$\frac{i\partial^n\hat{\varphi}}{(n-1)!} \longmapsto \hat{a}_{-n}|vac\rangle$

$i\partial^p\hat{\varphi}(z)|vac\rangle = (-1)^{p-1}\sum_n (n+1)\cdots(n+p-1)\hat{a}_n z^{-n-p}|vac\rangle \longrightarrow (p-1)!\,\hat{a}_{-p}|vac\rangle$

$\overbrace{\cdots -p-1}^{z^{>0}\to 0}\quad \underset{\text{survives!}}{-p}\quad \overbrace{-p+1 \cdots -1}^{\text{coeff}=0}\quad \overbrace{0\ \ 1\ \ 2 \cdots}^{a_n|vac\rangle=0} \longleftarrow n$

$:(i\partial\hat{\varphi})(i\partial\hat{\varphi}): \longmapsto \hat{a}_{-1}\hat{a}_{-1}|vac\rangle$

(#)

Generally: $\underbrace{:\left(\prod_{j=1}^{r}\frac{i}{(n_j-1)!}\partial^{n_j}\varphi\right)\left(\prod_{k=1}^{s}\frac{i}{(m_k-1)!}\bar{\partial}^{m_k}\varphi\right):}_{\text{normally-ordered diff. polynomial}} \longmapsto \hat{\bar{a}}_{-m_s}\cdots\hat{\bar{a}}_{-m_1}\hat{a}_{-n_r}\cdots\hat{a}_{-n_1}|vac\rangle$

$1\leq n_1\leq\cdots\leq n_r$, $1\leq m_1\leq\cdots\leq m_s$

- So, we recovered a huge part of the space of states - ~~everything~~ the full eigenspace $\Pi_0=0$ of $\hat{\Pi}_0$

Vertex operator

$$\boxed{\hat{V}_\alpha(z,\bar{z}) = :e^{i\alpha\hat{\varphi}(z,\bar{z})}:}$$

$\alpha\in\mathbb{R}$ a parameter

Recall:

$\hat{\varphi}(z,\bar{z}) = \hat{\varphi}_0 - i\hat{\Pi}_0\log(z\bar{z}) + \sum_{n\neq 0}\frac{i}{n}(\hat{a}_n z^{-n} + \hat{\bar{a}}_n\bar{z}^{-n})$

with $[\hat{a}_n,\hat{a}_m] = n\delta_{n,-m}\hat{\mathbb{1}} = [\hat{\bar{a}}_n,\hat{\bar{a}}_m]$, $[\hat{\varphi}_0,\hat{\Pi}_0] = i\hat{1}$

Let's try to act with $\hat{V}_\alpha(z,\bar{z})$ on $|vac\rangle$:

rep. of the Heisenberg subalgebra

$\hat{\Pi}_0:\ \psi(\Pi_0)\longmapsto \Pi_0\psi(\Pi_0)$

$\hat{\varphi}_0:\ \psi(\Pi_0)\longmapsto i\frac{\partial}{\partial\Pi_0}\psi(\Pi_0)$

$\Rightarrow e^{i\alpha\hat{\varphi}_0}:\ \psi(\Pi_0)\longmapsto \psi(\Pi_0-\alpha)$ - shift operator

in particular $\delta_{\Pi_0}\longmapsto\delta_{\Pi_0+\alpha}$

and $|vac\rangle\longmapsto$ ~~|α⟩~~ $|\Pi_0=\alpha\rangle$

$\hat{V}_\alpha(z,\bar{z})|vac\rangle = :e^{i\alpha\hat{\varphi}_0}e^{~~\alpha\hat{\Pi}_0\log~~}e^{i\alpha\sum_{n\neq 0}\frac{i}{n}(\hat{a}_n z^{-n}+\hat{\bar{a}}_n\bar{z}^{-n})}:|vac\rangle$

$= e^{i\alpha\sum_{n<0}\frac{i}{n}(\hat{a}_n z^{-n}+\hat{\bar{a}}_n\bar{z}^{-n})}e^{i\alpha\sum_{n>0}\frac{i}{n}(\hat{a}_n z^{-n}+\hat{\bar{a}}_n\bar{z}^{-n})}\underbrace{|\Pi_0=\alpha\rangle}_{\substack{\text{pseudo-vacuum}\\ \text{with zero-mode}\\ \text{momentum } \alpha}}$

$\hat{V}_\alpha(z,\bar{z}) = \left(e^{i\alpha\hat{\varphi}_0}e^{i\alpha\sum_{n<0}\frac{i}{n}(\hat{a}_n z^{-n}+\hat{\bar{a}}_n\bar{z}^{-n})}\right)\circ\left(e^{+\alpha\hat{\Pi}_0\log z\bar{z}}e^{i\alpha\sum_{n>0}\cdots}\right)$

In particular, $\boxed{\lim_{z\to 0}\hat{V}_\alpha(z,\bar{z})|vac\rangle = |\Pi_0=\alpha\rangle}$

* taking $\left(\underset{\text{at } z,\bar{z}}{\text{diff. polynomials in } \varphi\ (\#)} \times V_\alpha(z,\bar{z})\right)|vac\rangle$, we recover as $\lim_{z\to 0}$ ~~all~~ the entire $\mathcal{H}$.

[Exercise!]

Properties of operators $\hat{V}_\alpha$

① V_α is a primary field with $\left(h=\frac{\alpha^2}{2}, \bar{h}=\frac{\alpha^2}{2}\right)$ ~~from~~, i.e. $R\hat{T}(w)\hat{V}_\alpha(z,\bar{z}) \approx \frac{\frac{\alpha^2}{2}V_\alpha(z,\bar{z})}{(w-z)^2} + \frac{\partial V_\alpha(z,\bar{z})}{w-z} + \text{reg}$

$R\hat{\bar{T}}(\bar{w})\hat{V}_\alpha(z,\bar{z}) \approx$ similar

② $\langle V_\alpha(z,\bar{z})\,V_\beta(w,\bar{w})\rangle = \begin{cases} |z-w|^{-2\alpha^2} & \text{if } \beta=-\alpha \\ 0 & \text{otherwise}\end{cases}$

more generally $\left\langle\prod_{k=1}^{n}V_{\alpha_k}(z_k,\bar{z}_k)\right\rangle = \begin{cases}\prod_{1\leq j<k\leq n}|z_j-z_k|^{2\alpha_j\alpha_k} & \text{if } \sum\alpha_k=0 \\ 0 & \text{otherwise}\end{cases}$

③ $R\hat{V}_\alpha(z,\bar{z})V_\beta(w,\bar{w}) \sim |z-w|^{2\alpha\beta}V_{\alpha+\beta}(w,\bar{w}) + \text{less singular terms}$

④ $i\partial\varphi(z)\ V_\alpha(w,\bar{w}) \sim \frac{\alpha}{z-w}V_\alpha(w,\bar{w}) + \text{reg}$

How to compute $\langle V_\alpha(z,\bar z)\, V_\beta(w,\bar w)\rangle$?

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One possibility: let $|z|>|w|$

$$R\,\hat V_\alpha(z,\bar z)\,\hat V_\beta(w,\bar w) = e^{i\alpha\hat\varphi_0+\alpha\sum_{n<0}\frac{1}{n}(\hat a_n z^{-n}+\bar a_n\bar z^{-n})}\circ \underbrace{e^{\alpha\hat\pi_0\log z\bar z-\alpha\sum_{n>0}\frac{1}{n}(\hat a_n z^{-n}+\hat{\bar a}_n\bar z^{-n})}\circ e^{i\beta\hat\varphi_0-\beta\sum_{m<0}\frac{1}{m}(\hat a_m w^{-m}+\hat{\bar a}_m\bar w^{-m})}}_{\mathrm{I}\circ\mathrm{II}}\circ e^{\beta\hat\pi_0\log w\bar w-\beta\sum_{m>0}\frac{1}{m}(a_m\bar w_{(c)}}$$

$$\mathrm{I}\circ\mathrm{II} = \mathrm{II}\circ\mathrm{I}\cdot e^{\alpha\beta\log z\bar z-\alpha\beta\underbrace{\sum_{n>0}\frac{1}{n}\left(\left(\frac{w}{z}\right)^n+\left(\frac{\bar w}{\bar z}\right)^n\right)}_{-\log(1-\frac{w}{z})-\log(1-\frac{\bar w}{\bar z})}} = \mathrm{II}\circ\mathrm{I}\cdot|z-w|^{2\alpha\beta}$$

using BCH:

$$e^A\circ e^B \underset{\substack{\uparrow\\ \text{for } [A,B] \text{ central}}}{=} e^{A+B+\frac{1}{2}[A,B]} = e^B\circ e^A\cdot e^{[A,B]}$$

So: $\hat V_\alpha(z,\bar z)\,\hat V_\beta(w,\bar w) = \underbrace{:\hat V_\alpha(z,\bar z)\,\hat V_\beta(w,\bar w):}_{\text{contains } e^{i(\alpha+\beta)\hat\varphi_0}}\cdot|z-w|^{2\alpha\beta}$ $\;\Rightarrow\;$ $\langle V_\alpha(z,\bar z)V_\beta(w,\bar w)\rangle = |z-w|^{2\alpha\beta}$

– its VEV $= 0$ if $\alpha+\beta\neq 0$
$= 1$ if $\alpha+\beta=0$

Primary fields [in a general CFT] on $\mathbb{C}$ – in BPZ picture]

Belavin Polyakov Zamolodchikov

- field $\Phi(w,\bar w)\in \underbrace{\mathcal{H}_w}_{\text{space of loc. fields at } w}$ is $(h,\bar h)$-primary, if it satisfies the OPEs

$$\begin{cases} T(z)\Phi(w,\bar w)\sim \frac{\textcircled{h}}{(z-w)^2}\Phi(w,\bar w)+\frac{1}{z-w}\partial\Phi(w,\bar w)+\text{reg}\\ \bar T(\bar z)\Phi(w,\bar w)\sim\frac{\textcircled{$\bar h$}}{(\bar z-\bar w)^2}\Phi+\frac{1}{\bar z-\bar w}\bar\partial\Phi+\text{reg}\end{cases}$$

⇝ Transformation property (action of a c.v.f.):

$$\delta_{\overset{\text{holom. v.f.}}{\varepsilon\partial}}\hat\Phi(w,\bar w) := [\rho(\varepsilon\partial),\hat\Phi(w,\bar w)]$$
$$= -\frac{1}{2\pi i}\oint dz\, R\,\hat T(z)\hat\Phi(w,\bar w)\,\varepsilon(z) = -h\,\partial\varepsilon(w)\hat\Phi(w,\bar w)-\varepsilon(w)\,\partial\hat\Phi(w,\bar w)$$

full infinitesimal transf. under c.v.f.:
$$\delta_{\varepsilon\partial+\bar\varepsilon\bar\partial}\Phi(w,\bar w) = -\varepsilon\cdot\partial\Phi-\bar\varepsilon\cdot\bar\partial\Phi-h\cdot\partial\varepsilon\Phi-\bar h\cdot\bar\partial\bar\varepsilon\Phi$$

⇝ action by a finite conformal map $z\mapsto w(z)$ <change of local coordinate>:
$$\Phi_{(z)}(z,\bar z)\longmapsto\Phi_{(w)}(w,\bar w)=\Phi_{(z)}(z,\bar z)\left(\frac{\partial z}{\partial w}\right)^h\left(\frac{\partial\bar z}{\partial\bar w}\right)^{\bar h}$$

or: $\Phi_{(z)}(z,\bar z)\to\left(\frac{\partial w}{\partial z}\right)^h\left(\frac{\partial\bar w}{\partial\bar z}\right)^{\bar h}\Phi_{(w)}(w(z),\bar w(z))$

Or: $\boxed{\Phi(z,\bar z)(dz)^h(d\bar z)^{\bar h}}$ is a coordinate-invariant object

Local Virasoro action at puncture z_0:
$$\rho_{(z_0)}\left(\varepsilon(z)\frac{\partial}{\partial z}\right)\circ\underbrace{\Phi(z_0,\bar z_0)}_{\in\mathcal{H}_{z_0}} := -\frac{1}{2\pi i}\oint dz\,\varepsilon(z)T(z)\Phi(z_0,\bar z_0)$$

generators: $L_n^{(z_0)}\circ\Phi(z_0,\bar z_0)=\frac{1}{2\pi i}\oint dz\,(z-z_0)^{n+1}T(z)\Phi(z_0,\bar z_0)$

Equivalently: $T(z)\Phi(z_0,\bar z_0)=\sum_{n\in\mathbb{Z}}(z-z_0)^{-n-2}\left(L_n^{(z_0)}\circ\Phi(z_0,\bar z_0)\right)$

- Φ is $(h,\bar h)$-primary $\Leftrightarrow$ $\begin{cases}L^{(z_0)}_{>0}\Phi(z_0,\bar z_0)=0=\bar L^{(z_0)}_{>0}\Phi(z_0,\bar z_0)\\ L_0\Phi=h\Phi\quad,\quad\bar L_0\Phi=\bar h\Phi\end{cases}$

$S(w,z):=\frac{\partial_z^3 w}{\partial_z w}-\frac{3}{2}\left(\frac{\partial_z^2 w}{\partial_z w}\right)^2$ – Schwarzian derivative

transformation law for T:

finite ver. $z\mapsto w(z)$

$$TT\text{ OPE}\Rightarrow\delta_\varepsilon T=-\varepsilon\partial T-2\partial\varepsilon\,T-\frac{c}{12}\partial^3\varepsilon \rightsquigarrow T_{(z)}(z)\mapsto T_{(w)}(w)=\left(\frac{\partial w}{\partial z}\right)^{-2}\left(T(z)-\frac{c}{12}S(w,z)\right)=\left(\frac{\partial w}{\partial z}\right)^{-2}T(z)+\frac{c}{12}S(z,w)$$