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1

How to compute $\langle V_\alpha(z,\bar z)\, V_\beta(w,\bar w)\rangle$?

One possibility: let $|z|>|w|$

$$\mathcal{R}\,\hat V_\alpha(z,\bar z)\,\hat V_\beta(w,\bar w) = e^{i\alpha\hat\varphi_0+\alpha\sum_{n<0}\frac{1}{n}(\hat a_n z^{-n}+\hat{\bar a}_n\bar z^{-n})}\circ \underbrace{e^{\alpha\hat\pi_0\log z\bar z-\alpha\sum_{n>0}\frac{1}{n}(\hat a_n z^{-n}+\hat{\bar a}_n\bar z^{-n})}\circ e^{i\beta\hat\varphi_0-\beta\sum_{m<0}\frac{1}{m}(\hat a_m w^{-m}+\hat{\bar a}_m\bar w^{-m})}}_{\mathrm{I}\circ\mathrm{II}}\circ e^{\beta\hat\pi_0\log w\bar w-\beta\sum_{m>0}\frac{1}{m}(a_m\bar w\ldots)}$$

$$\mathrm{I}\circ\mathrm{II} = \mathrm{II}\circ\mathrm{I}\cdot e^{\alpha\beta\log z\bar z-\alpha\beta\underbrace{\sum_{n>0}\frac{1}{n}\left(\left(\frac{w}{z}\right)^n+\left(\frac{\bar w}{\bar z}\right)^n\right)}_{-\log(1-\frac{w}{z})-\log(1-\frac{\bar w}{\bar z})}} = \mathrm{II}\circ\mathrm{I}\cdot|z-w|^{2\alpha\beta}$$

using BCH:

$$e^A\circ e^B \underset{\substack{\uparrow\\ \text{for } [A,B] \text{ central}}}{=} e^{A+B+\frac{1}{2}[A,B]} = e^B\circ e^A\cdot e^{[A,B]}$$

So: $$\hat V_\alpha(z,\bar z)\,\hat V_\beta(w,\bar w) = \underbrace{:\hat V_\alpha(z,\bar z)\,\hat V_\beta(w,\bar w):}_{\text{contains } e^{i(\alpha+\beta)\hat\varphi_0}}\cdot|z-w|^{2\alpha\beta} \rightarrow \langle V_\alpha(z,\bar z)V_\beta(w,\bar w)\rangle = |z-w|^{2\alpha\beta}$$

– its VEV $= 0$ if $\alpha+\beta\neq 0$
$= 1$ if $\alpha+\beta=0$

Primary fields [in a general CFT] on $\mathbb{C}$ – in BPZ picture]

Belavin Polyakov Zamolodchikov

– field $\Phi(w,\bar w)\in\underset{\text{space of loc fields at } w}{\mathcal{H}_w}$ is $(h,\bar h)$-primary, if it satisfies the OPEs

$$\begin{cases} T(z)\Phi(w,\bar w)\sim\frac{h}{(z-w)^2}\Phi(w,\bar w)+\frac{1}{z-w}\partial\Phi(w,\bar w)+\text{reg}\\ \bar T(\bar z)\Phi(w,\bar w)\sim\frac{\bar h}{(\bar z-\bar w)^2}\Phi+\frac{1}{\bar z-\bar w}\bar\partial\Phi+\text{reg}\end{cases}$$

→ Transformation property (action of a c.v.f.):

$$\delta_{\overset{\text{holom. v.f.}}{\varepsilon\partial}}\hat\Phi(w,\bar w):=[\rho(\varepsilon\partial),\hat\Phi(w,\bar w)]$$
$$= -\frac{1}{2\pi i}\oint dz\,\mathcal{R}\,\hat T(z)\hat\Phi(w,\bar w)\,\varepsilon(z) = -h\,\partial\varepsilon(w)\hat\Phi(w,\bar w)-\varepsilon(w)\,\partial\hat\Phi(w,\bar w)$$

deform

full infinitesimal transf. under c.v.f.: $$\delta_{\varepsilon\partial+\bar\varepsilon\bar\partial}\Phi(w,\bar w) = -\varepsilon\cdot\partial\Phi-\bar\varepsilon\cdot\bar\partial\Phi-h\cdot\partial\varepsilon\Phi-\bar h\cdot\bar\partial\bar\varepsilon\,\Phi$$

⤳ action by a finite conformal map $z\longmapsto w(z)$ <change of local coordinate>:

$$\Phi_{(z)}(z,\bar z)\longmapsto\Phi_{(w)}(w,\bar w) = \Phi_{(z)}(z,\bar z)\left(\frac{\partial z}{\partial w}\right)^h\left(\frac{\partial\bar z}{\partial\bar w}\right)^{\bar h}$$

or: $\Phi_{(z)}(z,\bar z)\to\left(\frac{\partial w}{\partial z}\right)^h\left(\frac{\partial\bar w}{\partial\bar z}\right)^{\bar h}\Phi_{(w)}(w(z),\bar w(\bar z))$

Or: $\boxed{\Phi(z,\bar z)(dz)^h(d\bar z)^{\bar h}}$ is a coordinate-invariant object

Local Virasoro action at puncture z_0:

$$\rho_{(z_0)}\left(\varepsilon(z)\frac{\partial}{\partial z}\right)\circ\underbrace{\Phi(z_0,\bar z_0)}_{\in\mathcal{H}_{z_0}} := -\frac{1}{2\pi i}\oint dz\,\varepsilon(z)T(z)\,\Phi(z_0,\bar z_0)$$

generators: $$L_n^{(z_0)}\circ\Phi(z_0,\bar z_0) = \frac{1}{2\pi i}\oint dz\,(z-z_0)^{n+1}T(z)\,\Phi(z_0,\bar z_0)$$

Equivalently: $$T(z)\Phi(z_0,\bar z_0) = \sum_{n\in\mathbb{Z}}(z-z_0)^{-n-2}\left(L_n^{(z_0)}\circ\Phi(z_0,\bar z_0)\right)$$

$$\boxed{T(z_0)=L_{-2}^{(z_0)}\circ\mathbb{1}_{(z_0)}}$$

• Φ is $(h,\bar h)$-primary $\Leftrightarrow\begin{cases}L_{>0}^{(z_0)}\Phi(z_0,\bar z_0)=0=\bar L_{>0}^{(z_0)}\Phi(z_0,\bar z_0)\\ L_0\Phi=h\Phi\quad,\quad\bar L_0\Phi=\bar h\,\Phi\end{cases}$

$L_{-1}^{(z_0)}\Phi=\partial\Phi$ for any Φ

$$S(w,z):=\frac{\partial_z^3 w}{\partial_z w}-\frac{3}{2}\left(\frac{\partial_z^2 w}{\partial_z w}\right)^2$$ – Schwarzian derivative

transformation law for T:

finite ver. $z\mapsto w(z)$

$$TT\text{ OPE}\Rightarrow\delta_\varepsilon T=-\varepsilon\,\partial T-2\partial\varepsilon\cdot T-\frac{c}{12}\partial^3\varepsilon\rightsquigarrow T_{(z)}(z)\mapsto T_{(w)}(w)=\left(\frac{\partial w}{\partial z}\right)^{-2}\left(T(z)-\frac{c}{12}S(w,z)\right)=\left(\frac{\partial z}{\partial z}\right)^{-2}T(z)+\frac{c}{12}S(z$$

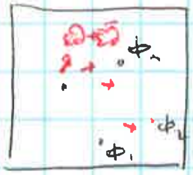
$$T_z(z)=\left(\frac{\partial w}{\partial z}\right)^2T_w(w)+\frac{c}{12}S(w,z)$$

3/27/19
2

• Correlation functions of primary fields <Inaugural CFT>

WARD Identity

$$\underbrace{\delta_{\epsilon\partial+\bar\epsilon\bar\partial}}_{\text{c.v.f.}}\langle\Phi_1(z_1,\bar z_1)\dots\Phi_n(z_n,\bar z_n)\rangle = 0 \quad \leftarrow \text{diff. equation on the correlator for each } \overset{\text{(global)}}{\text{c.v.f.}}$$



$\Phi_k\in\mathcal{H}_k^{(z_k)}$ space of local fields at z_k

$\Rightarrow$ for $f: z\mapsto \overset{\overset{w(z)}{\shortparallel}}{f(z)}$ conf. map

global conformal invariance of the correlator

$$\langle\Phi_1(z_1,\bar z_1)\cdots\Phi_n(z_n,\bar z_n)\rangle=\langle\Phi_1(f(z_1),\bar f(\bar z_1))\left(\frac{\partial f}{\partial z}\Big|_{z_1}\right)^{h}\left(\frac{\partial\bar f}{\partial\bar z}\Big|_{\bar z_1}\right)^{\bar h}\cdots\Phi_n(f(z_n),\bar f(\bar z_n))\left(\frac{\partial f}{\partial z}\Big|_{z_n}\right)^{h}\left(\frac{\partial \bar f}{\partial \bar z}\Big|_{\bar z_n}\right)^{\bar h}\rangle$$

Or better:

$$\langle\Phi_1(z_1,\bar z_1)(dz_1)^{h_1}(d\bar z_1)^{\bar h_1}\cdots\Phi_n(z_n,\bar z_n)(dz_n)^{h_n}(d\bar z_n)^{\bar h_n}\rangle$$
$$=\langle\Phi_1(w_1,\bar w_1)(dw_1)^{h_1}(d\bar w_1)^{\bar h_1}\cdots\Phi_n(w_n,\bar w_n)(dw_n)^{h_n}(d\bar w_n)^{\bar h_n}\rangle$$

4/1/19
1

In other words, the correlator of primary fields is a Möbius-invariant section

$$\langle\Phi_1(z_1,\bar z_1)(dz_1)^{h_1}(d\bar z_1)^{\bar h_1}\cdots\rangle\in\Gamma\Big(\underset{\text{open config. space of } n \text{ points } z_1\dots z_n}{C_n(\mathbb{CP}^1)},\ \underline{(K^{\otimes h_1}\otimes\bar K^{\otimes\bar h_1})\boxtimes\cdots\boxtimes(K^{\otimes h_n}\otimes\bar K^{\otimes\bar h_n})}\Big)^{PSL_2(\mathbb{C})}$$

$K = (T^{1,0})^*\mathbb{CP}^1$ - bundle of (1,0)-forms

$\bar K = (T^{0,1})^*\mathbb{CP}^1$ - bundle of (0,1)-forms

Theorem (classification of Möbius-invariant sections of line bundles over config. spaces of points on $\mathbb{CP}^1$)

uniquely determined, up to const

case
- $n=1$: $\langle\Phi(z,\bar z)(dz)^h(d\bar z)^{\bar h}\rangle=\begin{cases}\text{const}, & \text{if } h=0=\bar h\\ 0, & \text{otherwise}\end{cases}$
- $n=2$: $\langle\Phi_1(z,\bar z)(dz)^{h_1}(d\bar z)^{\bar h_1}\Phi_2(w,\bar w)(dw)^{h_2}(d\bar w)^{\bar h_2}\rangle=\begin{cases}C\dfrac{(dz)^h(dw)^h}{(z-w)^{2h}}\dfrac{(d\bar z)^{\bar h}(d\bar w)^{\bar h}}{(\bar z-\bar w)^{2\bar h}} & \text{if } \begin{cases}h_1=h_2=:h\\ \bar h_1=\bar h_2=:\bar h\end{cases}\\ 0 & \text{otherwise}\end{cases}$
- $n=3$: $\langle\Phi_1(z_1,\bar z_1)(dz_1)^{h_1}(d\bar z_1)^{\bar h_1}\cdots\Phi_3(z_3,\bar z_3)(dz_3)^{h_3}(d\bar z_3)^{\bar h_3}\rangle=C\prod_{1\le i<j\le 3}\left(\dfrac{dz_i\,dz_j}{(z_i-z_j)^2}\right)^{\alpha_{ij}}\left(\dfrac{d\bar z_i\,d\bar z_j}{(\bar z_i-\bar z_j)^2}\right)^{\bar\alpha_{ij}}$

  where $\boxed{\alpha_{ij}=\alpha_{ji},\ \alpha_{ii}=0,\ \sum_j\alpha_{ij}=h_i}$ $\Rightarrow$ (*)

  $\Rightarrow \alpha_{12}+\alpha_{13}=h_1$
  $\alpha_{12}+\alpha_{23}=h_2$ $\Rightarrow \alpha_{12}=\frac12(h_1+h_2-h_3)$
  $\alpha_{13}+\alpha_{23}=h_3$ $\quad\alpha_{13}=\frac12(h_1+h_3-h_2)$
  $\alpha_{23}=\frac12(h_2+h_3-h_1)$
  and similarly for $\bar\alpha_{ij}$
- $n=4$: $\langle\Phi_1(dz_1)^{h_1}(d\bar z_1)^{\bar h_1}\cdots\Phi_4(dz_4)^{h_4}(d\bar z_4)^{\bar h_4}\rangle=$

  $$=\prod_{1\le i<j\le 4}\left(\frac{dz_i\,dz_j}{(z_i-z_j)^2}\right)^{\alpha_{ij}}\left(\frac{d\bar z_i\,d\bar z_j}{(\bar z_i-\bar z_j)^2}\right)^{\bar\alpha_{ij}}\cdot\mathcal{F}(\lambda)$$

  any function

  cross-ratio of $z_1\dots z_4$
- arbitrary $n\ge4$
  - similar, contains an arbitrary function $\mathcal{F}(\lambda_1\dots\lambda_{n-3})$ of $n-3$ independent cross-ratios

Remark/recall:

$\dfrac{dz\,dw}{(z-w)^2}\in\Omega^2(C_2(\mathbb{CP}^1))$ - a Möbius-invariant holomorphic 2-form

Rem (one of) solution to (*):

$$\begin{cases}\alpha_{ij}=\dfrac{1}{n-2}\left(h_i+h_j-\dfrac{1}{n-1}\sum_{k=1}^n h_k\right), & i\ne j\\ \alpha_{ii}=0\end{cases}$$

5/1/19
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3/27/19
3

WARD Identity:

$$\delta_{\varepsilon\partial+\bar\varepsilon\bar\partial}\langle\phi_1(z_1,\bar z_1)\ldots\phi_n(z_n,\bar z_n)\rangle := \sum_{k=1}^{n}\langle\phi_1(z_1,\bar z_1)\ldots \underbrace{\rho(\varepsilon\partial+\bar\varepsilon\bar\partial)\circ\phi_k(z_k,\bar z_k)}\cdots\phi_n(z_n,\bar z_n)\rangle = 0 \qquad (*)$$

~~$\frac{1}{2\pi i}\oint_{\circlearrowleft}(dw\,(T(w)\varepsilon(w) + \bar T(\bar w)\bar\varepsilon(\bar w))\,\phi_k(z_k,\bar z_k))$~~

$$-\frac{1}{2\pi i}\oint_{\circlearrowleft_{z_k}} dw\,\varepsilon(w)T(w)\phi_k(z_k,\bar z_k) + \frac{1}{2\pi i}\oint_{\circlearrowleft} d\bar w\,\bar\varepsilon(\bar w)\bar T(\bar w)\phi_k(z_k,\bar z_k)$$

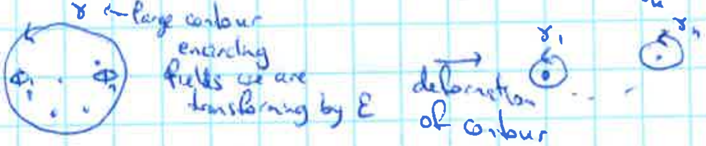
Apply (*) to

$\varepsilon(z)\partial = -\frac{1}{z-z_0}\frac{\partial}{\partial z}$ - simple pole hol. v.f. at z_0
(+ insert field $\mathbb{1}$ at z_0 in the correlator)

$$\rightarrow \langle T(z_0)\phi_1(z_1,\bar z_1)\ldots\phi_n(z_n,\bar z_n)\rangle + \sum_{k=1}^{n}\langle \mathbb{1}_{z_0}\phi_1(z_1,\bar z_1)\ldots\left(\frac{1}{z_k-z_0}\frac{\partial}{\partial z_k} - \frac{h_k}{(z_k-z_0)^2}\right)\phi_k(z_k,\bar z_k)\ldots\phi_n(z_n,\bar z_n)\rangle = 0$$

Or: $$\boxed{\langle T(z_0)\phi_1(z_1,\bar z_1)\ldots\phi_n(z_n,\bar z_n)\rangle = \sum_{k=1}^{n}\left(\frac{h_k}{(z_k-z_0)^2} - \frac{1}{z_k-z_0}\frac{\partial}{\partial z_k}\right)\circ\langle\phi_1(z_1,\bar z_1)\ldots\phi_n(z_n,\bar z_n)\rangle}$$

- (*): $\delta_{\varepsilon\partial}\langle\phi_1\ldots\phi_n\rangle = \langle -\frac{1}{2\pi i}\oint_{\gamma} dw\,\varepsilon(w)T(w)\;\phi_1\ldots\phi_n\rangle = \sum_k\langle\phi_1\ldots -\frac{1}{2\pi i}\oint_{\gamma_k} dw\,\varepsilon(w)T(w)\phi_k(z_k,\bar z_k)\ldots\phi_n\rangle$



Space of fields at z_0, $\mathcal{H}_{z_0}$ is equipped with action of "local" Virasoro,

$$L_n^{(z_0)}: \phi(z_0,\bar z_0) \longmapsto -\frac{1}{2\pi i}\oint_{\circlearrowleft_{z_0}} dz\,(z-z_0)^{n+1}T(z)\phi(z_0,\bar z_0)$$

equivalently: $$T(z)\phi(z_0,\bar z_0) = \sum_{n\in\mathbb{Z}}(z-z_0)^{-n-2}L_n^{(z_0)}\phi(z_0,\bar z_0) = \ldots + \frac{L_1\phi_{z_0}}{(z-z_0)^3} + \frac{L_0\phi_{z_0}}{(z-z_0)^2} + \frac{L_{-1}\phi_{z_0}}{z-z_0} + \text{reg.}$$

& similarly for $\bar T, \bar L_n$

field ϕ is $\underbrace{(h,\bar h)}_{\text{hol/anti-hol conf. dimension}}$ - primary iff $\begin{array}{l} L_{>0}\phi = 0, \quad \bar L_{>0}\phi = 0 \\ L_0\phi = h\phi \quad \bar L_0\phi = \bar h\phi \end{array}$, Thus $T(z)\phi(z_0,\bar z_0) = \frac{h\phi_{z_0}}{(z-z_0)^2} + \frac{\overset{\text{"}L_{-1}\phi}{\partial\phi_{z_0}}}{z-z_0} + \text{reg}$

- $\mathcal{H} = \bigoplus_{\alpha} \underbrace{\mathcal{H}^{(\phi_\alpha)}}_{}$

  sum over species of primary fields

  $\text{Span}\{\underbrace{L_{-k_r}\ldots L_{-k_1}\bar L_{-\ell_s}\ldots\bar L_{-\ell_1}\phi_\alpha}\}_{\substack{1\le k_1\le\ldots\le k_r\\ 1\le\ell_1\le\ldots\le\ell_s}}$ ← Verma module for Vir⊕$\overline{\text{Vir}}$

  "descendant fields" ⟨from the primary field ϕ_α⟩

  [Sometimes, for unitarity, we want to quotient out a submodule!]

- $L_{-1}\phi = \partial\phi$ for <u>any</u> field ϕ (not necessarily primary)
  $\bar L_{-1}\phi = \bar\partial\phi$

ϕ (not necessarily primary) is of conf. dim $(h,\bar h)$ if $L_0\phi = h\phi$, $\bar L_0\phi = \bar h\phi$.

· $\mathcal{H}$ is bi-graded by $(h,\bar h)$

· L_{-n} changes h by $+n$

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- $\mathbb{1}$ - distinguished primary field

$$\boxed{L_{-1}\mathbb{1} = 0 = \bar{L}_{-1}\mathbb{1}}$$

can be seen from $\langle \mathbb{1}_{z_0}\Phi_{z_1}\cdots\Phi_{z_n}\rangle = \langle \Phi_{z_1}\cdots\Phi_{z_n}\rangle$ - independent of z_0

also $L_{-1}\mathbb{1}$ would have to be a vector of zero norm!

$$\boxed{L_{-2}\mathbb{1} = T}$$

- Correlators of descendants can be expressed via corr. of primary fields. (from the Ward identity)

Ex: $$\langle \underset{\substack{\uparrow\\ \text{action of } -(z-z_0)^{-k+1}\frac{\partial}{\partial z} = \varepsilon\frac{\partial}{\partial z}}}{L_{-k}}\Phi_0(z_0,\bar{z}_0)\,\Phi_1(z_1,\bar{z}_1)\cdots\Phi_n(z_n,\bar{z}_n)\rangle = -\sum_{j=1}^{n}\Big(\underbrace{(z_j-z_0)^{-k+1}\frac{\partial}{\partial z_j} + (-k+1)(z_j-z_0)^{-k}h_j}_{\text{from Taylor expansion of } \varepsilon(z)\frac{\partial}{\partial z} \text{ near } z_j}\Big)\circ\langle\Phi_0(z_0,\bar{z}_0)\cdots\Phi_n(z_n,\bar{z}_n)\rangle$$

Terms $O(z^2\frac{\partial}{\partial z})$ vanish when acting on a primary field!

- transformation law for T & Schwar~~t~~zian derivative

$$TT \text{ OPE} \Longrightarrow \quad \delta_{\varepsilon\frac{\partial}{\partial z}}T = \underbrace{-\varepsilon\partial T - 2\partial\varepsilon T}_{\substack{\text{transf. of a primary}\\ \text{field of } h=2}}\underbrace{-\frac{c}{12}\partial^3\varepsilon}_{\substack{\text{correction}\\ \text{due to central extension}}}$$

finite version:

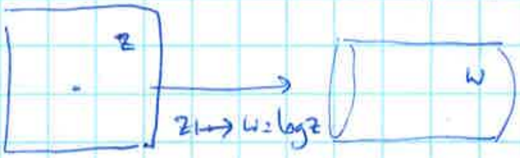
for $f: z \longmapsto w$ a holom. map,

$$T_{(z)}(z) \longrightarrow \left(\frac{\partial w}{\partial z}\right)^2 T_{(w)}(w) + \frac{c}{12}S(w,z)$$

$$S(w,z) = \frac{\partial_z^3 w}{\partial_z w} - \frac{3}{2}\left(\frac{\partial_z^2 w}{\partial_z w}\right)^2$$ - Schwar~~t~~zian derivative

- $S(z+\lambda\varepsilon(z), z) = \lambda\varepsilon'''(z) + O(\lambda^2)$

* S vanishes for Möbius transformations
* S satisfies a chain-like property $S(f\circ g) = (S(f)\circ g)(g')^2 + S(g)$
* S can be interpreted as a ~~g~~ group 1-cocycle in $H^1(\mathrm{Diff}(S^1), \underbrace{F_2(S^1)}_{\text{densities of weight } 2})$
* Ex:



$S\{w,z\} = \frac{1}{2z^2}$ ← Exercise!

Also: $S(z,w) = -\frac{1}{2}$

Thus, $$T_{(z)}(z) \longrightarrow \frac{1}{z^2}\left(T_{(w)}(w) + \boxed{\frac{c}{24}}\right)$$

In particular, if $\langle T(z)\rangle_{\mathbb{C}} = 0$ on the plane, then $\langle T(w)\rangle_{\text{cylinder}} = -\frac{c}{24}$ ← physical buzzword "Casimir energy" associated to the periodic boundary conditions

$\rightsquigarrow -\frac{c}{24R^2}$ ← radius

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Aside: vertex $\overset{\text{chiral algebra}}{\overset{\shortparallel}{\text{algebras}}}$ (Borcherds ⇝ V. Kac → ...)

Def a vertex algebra is

- a vector space V over $\mathbb{C}$, • a "vacuum vector" ~~to~~ ~~[illegible]~~ $\mathbb{1} \in V$ or $|vac\rangle$
- "translation operator" $\tau \in \operatorname{End} V$

usual notation T, but we don't want to confuse with stress-energy

- "state-field correspondence" $Y(-,z) : V \longrightarrow \operatorname{End} V[[z, z^{-1}]]$ ← formal Laurent series

$$A \longmapsto Y(A,z) = \sum_{n\in\mathbb{Z}} \underbrace{A_{(n)}}_{\in \operatorname{End} V} z^{-n-1}$$

Axioms

- (vacuum axiom) $Y(|vac\rangle, z) = Id_V$

  $\underset{\text{no negative powers}}{Y(A,z)|vac\rangle \in V[[z]]}$, $\lim_{z\to 0} Y(A,z)|vac\rangle = A$

- (translation axiom) $\forall A \in V$, $[\tau, Y(A,z)] = \partial_z Y(A,z)$

  $\tau |vac\rangle = 0$

- (Locality) $\forall A,B$ formal distributions $Y(A,z)$, $Y(B,w)$ are "mutually local":

  $\exists N$ s.t. $(z-w)^N Y(A,z)Y(B,w) = (z-w)^N Y(B,w)Y(A,z)$

4/18/19

Equivalently: $\forall u \in V^*$, $v \in V$, $\langle u, Y(A,z)Y(B,w)v\rangle$ and $\langle u, Y(B,w)Y(A,z)v\rangle$ are expansions of same element ~~in~~ $F_{u,A,B,v} \in \mathbb{C}[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$ finite-order pole!

Rem it follows from Def that $\tau A = A_{(-2)}|vac\rangle$, $\frac{\tau^n}{n!} A = A_{(-n-1)}|vac\rangle$ ⇝ $Y(A,z)|vac\rangle = e^{z\tau} A$

Ex [trivial]: $V, \cdot, 1, \tau$ - unital comm. algebra with derivation τ

⇝ vertex alg. with $|vac\rangle = 1$, $Y(A,z) \circ B = e^{z\tau}(A) \cdot B$

~~~~~~~~~~~~~~~~

Vertex <u>operator</u> algebra - VA which is $\mathbb{Z}_{\geq 0}$-graded, $V = \bigoplus_{h\geq 0} V_h$; for $A \in V_{h_1}$, $B \in V_{h_2}$, $A_n(B) \in V_{h_1+h_2-n-1}$,

and has a "conformal vector" $\omega \in V_2$ s.t.

coeffs in $\underset{\underset{T(z)}{\shortparallel}}{Y(\omega,z)} = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ satisfy

- $L_0 A = h A$ for $A \in V_h$
- $L_{-1} = \tau$
- $[L_n, L_m] = \cdots$ ← Virasoro comm. rel.

$T(z)$ - stress-energy ~~energy-momentum~~ field

Ex (free boson / Heisenberg VOA): $V = \operatorname{Span}\{a_{-k_n} \cdots a_{-k_1}|vac\rangle\}_{1\leq k_1 \leq \cdots \leq k_n}$

- Fock space (Verma module) for Heisenberg Lie algebra $[a_n, a_m] = ~~[illegible]~~\ n\,\delta_{n,-m}\mathbb{1}$

$$Y(a_{-k_n} \cdots a_{-k_1}|vac\rangle, z) = \frac{1}{(k_1-1)! \cdots (k_n-1)!} :i\partial^{k_1}\varphi(z) \cdots i\partial^{k_n}\varphi(z):$$

where $i\partial\varphi(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$

$\omega = (a_{-1})^2|vac\rangle$

$Y(\omega,z) = -\frac{1}{2} :\partial\varphi(z)^2:$ $\tau = L_{-1} = \frac{1}{2}\sum_{n\in\mathbb{Z}} a_n a_{-1-n}$

Ex: $\langle vac| \underbrace{Y(a_{-1}|vac\rangle, z)}_{i\partial\varphi(z)} \underbrace{Y(a_{-1}|vac\rangle, w)}_{i\partial\varphi(w)} |vac\rangle$ $\in (\mathbb{C}[[z^{\pm 1}]])[[w^{\pm 1}]]$

$= \sum_{n>0} n\, z^{-n-1} w^{n-1}$ ← expansion first in w then in z of $\frac{1}{(z-w)^2}$

↙ exp in z then in w

$\sum_{m>0} m\, w^{-m-1} z^{m-1}$

$\shortparallel$

$\mathbb{C}[[w^{\pm 1}]][[z^{\pm 1}]] \ni \langle vac| Y(a_{-1}|vac\rangle, w)\, Y(a_{-1}|vac\rangle, z) |vac\rangle$

Ex (Virasoro VOA): $V = \operatorname{Span}\{L_{-k_n} \cdots L_{-k_1}|vac\rangle\}_{2\leq k_1 \leq \cdots \leq k_n}$

$$Y(L_{-k_n} \cdots L_{-k_1}|vac\rangle, z) = \frac{1}{(k_1-2)! \cdots (k_n-2)!} :\partial^{k_1-2}T(z) \cdots \partial^{k_n-2}T(z):$$

[illegible]

where $T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ stress-energy tensor
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CFT $^{\text{Spring}}_{2019}$ - 2nd notebook

4/8/19
2

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Free boson with values in S^1 (free boson "compactified" on a circle)

Classically: $S[\varphi] = \frac{1}{2}\int_\Sigma d\tau\, d\sigma\left((\partial_\tau\varphi)^2 + (\partial_\sigma\varphi)^2\right)$

$\varphi : \overset{\text{cylinder}}{\Sigma} \longrightarrow \underbrace{S^1}_{\mathbb{R}/2\pi r}$

$\varphi(\sigma+2\pi, \tau) = \varphi(\sigma,\tau) + 2\pi r \cdot \boxed{m}$

$m \in \mathbb{Z}$ - winding number

Space of fields $F = \mathrm{Map}(\Sigma, S^1) = \coprod_{m\in\mathbb{Z}} \mathrm{Map}_m(\Sigma, S^1)$

↑ maps with winding number m.

Configuration space: $X = \coprod_{m\in\mathbb{Z}} \underbrace{\mathrm{Map}_m(S^1, S^1)}_{X_m}$

$\varphi(\sigma) = \varphi_0 + m r\cdot\sigma + \sum_{n\neq 0}\varphi_n e^{in\sigma}$ in X_m sector

Hamiltonian: $H = \pi_0^2 + \underbrace{\left(\frac{mr}{2}\right)^2}_{} + \sum_{n\neq 0}\left(\pi_n\pi_{-n} + \frac{1}{4}n^2\varphi_n\varphi_{-n}\right)$

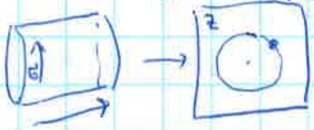
due to σ-dependence of 0-mode in φ

Canonical quantization

$\mathcal{H} = \bigoplus_{m\in\mathbb{Z}} \mathcal{H}_m$

states with winding number m

Heisenberg field: $\hat\varphi(z,\bar z) = \hat\varphi_0 - i\frac{\hat m r}{2}\log\frac{z}{\bar z} - i\hat\pi_0\log(z\bar z) + \sum_{n\neq 0}\frac{i}{n}\left(\hat a_n z^{-n} + \hat{\bar a}_n \bar z^{-n}\right)$



- since $\hat\varphi_0$ is defined mod $2\pi r\mathbb{Z}$, $\hat\pi_0$ has $\overset{\text{eigenvalue}}{\text{spectrum}}$ $\frac{1}{r}\mathbb{Z}$.

Indeed: in Schrödinger rep.,

$\mathcal{H}_{z.m.} = L^2(S^1) = \{\Psi(\varphi_0)$ - $2\pi r$-periodic: $\Psi(\varphi_0+2\pi r) = \Psi(\varphi_0)\}$

$\hat\varphi_0 : \Psi(\varphi_0) \longmapsto \varphi_0\Psi(\varphi_0)$

$\hat\pi_0 : \Psi(\varphi_0) \longmapsto -i\frac{\partial}{\partial\varphi_0}\Psi(\varphi_0)$

in the basis $\{\Psi_e = e^{\frac{ie\varphi_0}{r}}\}_{e\in\mathbb{Z}}$, $\hat\pi_0$ is diagonal, with eigenvalues $\frac{e}{r}$, $e\in\mathbb{Z}$.

Introduce $\hat e := r\hat\pi_0$. Then $\hat e$ has spectrum $\mathbb{Z}$.

Then: $\hat\varphi(z,\bar z) = \hat\varphi_0 - i\frac{\hat m r}{2}\log\frac{z}{\bar z} - i\frac{\hat e}{r}\log z\bar z + \sum_{n\neq 0}\frac{i}{n}\left(\hat a_n z^{-n} + \hat{\bar a}_n\bar z^{-n}\right)$

$i\partial\hat\varphi(z) = \sum_{n\in\mathbb{Z}}\hat a_n z^{-n-1}$, setting $\hat a_0 := \frac{\hat e}{r} + \frac{\hat m r}{2}$

$i\bar\partial\hat\varphi(\bar z) = \sum_{n\in\mathbb{Z}}\hat{\bar a}_n\bar z^{-n-1}$, setting $\hat{\bar a}_0 := \frac{\hat e}{r} - \frac{\hat m r}{2}$

~~total~~ Hamiltonian $\hat H = \hat L_0 + \hat{\bar L}_0 = \frac{1}{2}\sum_{n\in\mathbb{Z}}\left(:\hat a_n\hat a_{-n}: + :\hat{\bar a}_n\hat{\bar a}_{-n}:\right)$

total momentum $\hat P = \hat L_0 - \hat{\bar L}_0 = \frac{1}{2}\sum_{n\in\mathbb{Z}}\left(:\hat a_n\hat a_{-n}: - :\hat{\bar a}_n\hat{\bar a}_{-n}:\right)$

$\frac{\quad}{3}$ $\cancel{\frac{4/3/19}{4}}$

Space of states

$$\mathcal{H} = \bigoplus_{(e,m)\in\mathbb{Z}^2} V^{\mathrm{Heis}\oplus\overline{\mathrm{Heis}}}_{\underbrace{\left(\frac{e}{r}+\frac{mr}{2}\right.}_{=:\alpha_{e,m}},\underbrace{\left.\frac{e}{r}-\frac{mr}{2}\right)}_{=:\bar\alpha_{e,m}}} = \mathrm{Span}\left\{ \hat a_{-k_r}\cdots\hat a_{-k_1}\ \hat{\bar a}_{-\ell_s}\cdots\hat{\bar a}_{-\ell_1}\underbrace{|e,m\rangle} \right\}_{\substack{e,m\in\mathbb{Z}\\ 1\le k_1\le\cdots\le k_r\\ 1\le \ell_1\le\cdots\le\ell_s}}$$

eigenvalues for $\hat a_0, \hat{\bar a}_0$

pseudo vacuum

$$\hat H\,|e,m\rangle = \left(\frac{1}{2}\left(\frac{e}{r}+\frac{mr}{2}\right)^2+\frac{1}{2}\left(\frac{e}{r}-\frac{mr}{2}\right)^2\right)|e,m\rangle = \left(\left(\frac{e}{r}\right)^2+\left(\frac{mr}{2}\right)^2\right)|e,m\rangle$$

$$\hat P\,|e,m\rangle = \left(-\,''\,-\ -\ -\,''\,-\right)|e,m\rangle = e\cdot m\,|e,m\rangle$$

also:

$$\hat L_0\,|e,m\rangle = \overbrace{\frac{1}{2}\left(\frac{e}{r}+\frac{mr}{2}\right)^2}^{h_{e,m}}|e,m\rangle$$

$$\hat{\bar L}_0\,|e,m\rangle = \underbrace{\frac{1}{2}\left(\frac{e}{r}-\frac{mr}{2}\right)^2}_{\bar h_{e,m}}|e,m\rangle$$

Vertex operators

Introduce the operator $\hat\mu$ ("defined mod $2\pi\mathbb{Z}$") s.t. $[\hat\mu,\hat m]=i$ and commuting with everything else; then $[\hat m, e^{ik\hat\mu}]=k\,e^{ik\hat\mu}$ for $k\in\mathbb{Z}$

thus $e^{ik\hat\mu}: \mathcal{H}_{e,m}\longrightarrow\mathcal{H}_{e,m+k}$ - increases m by $+k$

likewise $[\hat\varphi_0, \underbrace{\hat\pi_0}_{``\frac{\hat e}{r}"}]=i \Rightarrow e^{i\frac{\ell\hat\varphi_0}{r}}: \mathcal{H}_{e,m}\longrightarrow\mathcal{H}_{e+\ell,m}$

Define the "chiral parts" of $\hat\varphi$:

$$\hat\chi(z) := \frac{1}{2}\hat\varphi_0+\frac{\hat\mu}{r} - i\underbrace{\left(\frac{\hat e}{r}+\frac{\hat m r}{2}\right)}_{\hat a_0}\log z + \sum_{n\neq 0}\frac{i}{n}\hat a_n z^{-n}$$

$$\hat{\bar\chi}(\bar z) := \frac{1}{2}\hat\varphi_0-\frac{\hat\mu}{r} - i\underbrace{\left(\frac{\hat e}{r}-\frac{\hat m r}{2}\right)}_{\hat{\bar a}_0}\log \bar z + \sum_{n\neq 0}\frac{i}{n}\hat{\bar a}_n \bar z^{-n}$$

so that $\hat\varphi(z,\bar z) = \hat\chi(z)+\hat{\bar\chi}(\bar z)$

Vertex operator: $$\hat V_{e,m}(z,\bar z) := \ :e^{i\left(\frac{e}{r}+\frac{mr}{2}\right)\hat\chi(z)}\,e^{i\left(\frac{e}{r}-\frac{mr}{2}\right)\hat{\bar\chi}(\bar z)}:$$

- $\hat V_{e,m}$ is <u>primary</u>, with $\left(h=\frac{1}{2}\left(\frac{e}{r}+\frac{mr}{2}\right)^2,\ \bar h = \frac{1}{2}\left(\frac{e}{r}-\frac{mr}{2}\right)\right)$ conf. dimension

- $\lim_{z\to 0}\hat V_{e,m}(z,\bar z)\underbrace{|\mathrm{vac}\rangle}_{|e=0,m=0\rangle} = |e,m\rangle$