CFT EXERCISES, 2/1/2019

1. Liouville theorem, step-by-step

(i) Write the equation $L_\epsilon g = \omega g$ of a conformal vector field $\epsilon = \epsilon^i \partial_j$ on $\mathbb{R}^{p,q}$ (equipped with the standard metric $g = \eta_{ij} dx^i dx^j$, with $\eta_{ij} = \mathrm{diag}(\underbrace{1,\ldots,1}_{p},\underbrace{-1,\ldots,-1}_{q})$) in components:[1]

$$\partial_i \epsilon_j + \partial_j \epsilon_i = \omega \eta_{ij} \tag{1}$$

(ii) Prove:

$$\partial_i \epsilon^i = \frac{n}{2}\omega \tag{2}$$

$$\Delta \epsilon_i = (1 - \frac{n}{2})\partial_i \omega \tag{3}$$

where $n = p + q$ the total dimension and $\Delta = \partial_i \partial^i = \eta^{ij}\partial_i \partial_j$ the Laplacian.

(iii) From (3) obtain:

$$\frac{1}{2}\eta_{ij}\Delta\omega = (1 - \frac{n}{2})\partial_i \partial_j \omega \tag{4}$$

$$(n-1)\Delta\omega = 0 \tag{5}$$

(iv) From (4), (5) show that, for $n \notin \{1, 2\}$,

$$\partial_i \partial_j \omega = 0 \tag{6}$$

I.e., ω is at most linear in coordinates x^i.

(v) Taking derivatives of (1), show that

$$\partial_i \partial_j \epsilon_k = \frac{1}{2}(\partial_i \omega\, \eta_{jk} + \partial_j \omega\, \eta_{ik} - \partial_k \omega\, \eta_{ij}) \tag{7}$$

(vi) From (6), (7) deduce that, for $n \notin \{1, 2\}$, we have

$$\partial_i \partial_j \partial_k \epsilon_l = 0 \tag{8}$$

I.e., ϵ is at most quadratic in coordinates x^i.

(vii) Assume the most general quadratic ansatz for ϵ and linear ansatz for ω,

$$\epsilon_i(x) = a_i + b_{ij}x^j + c_{ijk}x^j x^k \tag{9}$$

$$\omega(x) = 2\mu + 4\nu_i x^i \tag{10}$$

with $a_i, b_{ij}, c_{ijk}, \mu, \nu_i$ some coefficients, and see what constraints does one have on these coefficients from (1).

[1] For simplicity, do this exercise first for the positive signature case, $p = n$, $q = 0$. In particular, then one can forget about the distinction between upper and lower indices.

2. From complex to conformal structure on a surface

Given a complex structure J on a surface Σ, construct a metric g on it as follows. Choose some (nowhere vanishing and agreeing with the orientation) area form $\sigma \in \Omega^2(\Sigma)$. For $u, v \in T_x\Sigma$ a pair of tangent vectors at a point $x \in \Sigma$, set

$$g_x(u, v) := \sigma_x(u, Jv)$$

Show that:

(a) g is symmetric and positive-definite.
(b) Conformal class of g is independent of the choice of σ.
(c) This construction inverts the construction associating a complex structure to a conformal structure,

$$g/\sim \quad \to \quad \begin{array}{cccc} J: & T_x\Sigma & \to & T_x\Sigma \\ & u & \mapsto & v \end{array}$$

where v is the "counterclockwise 90-degree rotation" of u w.r.t. any metric g representing the conformal class (i.e. an orhtogonal vector of same length, with the pair (u, v) positively oriented).

3. Conformal extension of vector fields on a circle into the disk

Consider the vector fields on the unit circle

$$u_k = \cos(k\theta)\frac{\partial}{\partial\theta} \quad , \quad v_k = \sin(k\theta)\frac{\partial}{\partial\theta}$$

for $k \in \mathbb{Z}$. Show that these vector fields can be extended to conformal vector fields on the unit disk only for $k \in \{-1, 0, 1\}$.

4. Cross-ratio

Show that the expression

$$[z_1, z_2 : z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

– assigning a complex number to a quadruple of distinct points in $\mathbb{C}$ – is $PSL_2(\mathbb{C})$-invariant. I.e., prove that

$$[\alpha(z_1), \alpha(z_2) : \alpha(z_3), \alpha(z_4)]$$

for any Möbius transformation $\alpha \in PSL_2(\mathbb{C})$.