CFT EXERCISES, 2/15/2019

1. Euler-Lagrange equations in classical field theory

Find the Euler-Lagrange equations in the following cases.

(a) Non-free scalar field on a Riemannian manifold (M, g):

(1)
$$S[\phi] = \int_M \frac{1}{2} d\phi \wedge *d\phi + V(\phi) d\mathrm{vol}_g \quad = \int_M \sqrt{\det g}\, d^n x \left(\frac{1}{2} (g^{-1})^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right)$$

with $\phi \in C^\infty(M)$ the scalar field, $V(\phi) = \sum_{k=0}^{p} \frac{a_k}{k!} \phi^k$ a fixed polynomial of degree $p \geq 3$, and $d\mathrm{vol}_g = *1 = \sqrt{\det g}\, d^n x$ the metric volume form.

(b) Yang-Mills theory on (M, g):

(2)
$$S[A] = -\frac{1}{2} \int_M \mathrm{tr}\; F_A \wedge *F_A \qquad = -\frac{1}{4} \int \mathrm{tr}\; F_{\mu\nu} F^{\mu\nu} d\mathrm{vol}_g$$

where the field A is a connection in a fixed principal G-bundle P over M (with G a semi-simple Lie group) and $F_A = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu \in \Omega^2(M, \mathrm{ad}(P))$ is the curvature of the connection; tr is the trace in the adjoint representation of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$.

(c) Chern-Simons theory on a 3-manifold M:

(3)
$$S[A] = \int_M \mathrm{tr} \left(\frac{1}{2} A \wedge dA + \frac{1}{3} A \wedge A \wedge A \right)$$

with the field $A \in \Omega^1(M, \mathfrak{g})$ – the 1-form of a connection in a trivial principal bundle $M \times G \to M$.

2. Stress-energy tensor

Find the stress-energy tensor[1]

$$T^{\mu\nu} = -\frac{2}{\sqrt{\det g}} \frac{\delta S_{M,g}}{\delta g_{\mu\nu}}$$

for the scalar field theory (1). Then check explicitly the conservation property $\nabla_\mu T^{\mu\nu} \sim 0$ modulo Euler-Lagrange equations.

[1] For evaluating the variation w.r.t. the variation of metric, the following identities are useful: $\delta\sqrt{\det g} = \mathrm{tr}(g^{-1}\delta g)\sqrt{\det g}$ (prove this from $\det g = e^{\mathrm{tr} \log g}$) and $\delta(g^{-1})^{\mu\nu} = -(g^{-1})^{\mu\alpha}\, \delta g_{\alpha\beta}\, (g^{-1})^{\beta\nu}$ (or in index-free form: $\delta g^{-1} = -g^{-1}\, \delta g\, g^{-1}$ where r.h.s. is understood as a matrix product).

3. Behavior of Hodge star under Weyl transformations

Let $*_g$ be the Hodge star associated to a metric g on an n-dimensional manifold M. Show that, for $g' = \Omega \cdot g$, with Ω a positive function on M, and for α any p-form, one has

$$*_{g'}\alpha = \Omega^{\frac{n}{2}-p} \cdot *_g\alpha$$

Note that this implies, in particular, that Hodge star is Weyl-invariant when action on forms of degree $\frac{n}{2}$ (for n even).

4. Example of a Noether current for a mixed source-target symmetry

Consider the free massless scalar field on Euclidean $\mathbb{R}^n$, defined by the action

$$S[\phi] = \int \frac{1}{2} d\phi \wedge *d\phi = \int \frac{1}{2} \partial_\mu \phi \, \partial_\mu \phi \; d^n x$$

And consider the mixed source/target transformation – dilatation on $\mathbb{R}^n$ accompanied by a rescaling of the field value

$$x \mapsto x' = \beta x \quad , \quad \phi(x) \mapsto \phi'(x') = \beta^{-\frac{n}{2}+1}\phi(x)$$

with $\beta > 0$ the scaling parameter. Or, equivalently,

$$\phi(x) \mapsto \phi'(x) = \beta^{-\frac{n}{2}+1}\phi(\beta^{-1}x)$$

Show that this is a symmetry (maps solutions of the EL equation to solutions). Show that the corresponding infinitesimal symmetry changes the Lagrangian density by a term of form $d\Lambda$ – and find Λ. Finally, find the Noether current corresponding to the symmetry.