CFT EXERCISES, 3/8/2019

1. Möbius invariance of 2-point correlator

As we have seen in the class, for the free boson on the plane we have the correlation function $\langle \partial\phi(z)\partial\phi(w)\rangle = -\frac{1}{(z-w)^2}$. Multiplying this expression by $dz \wedge dw$, we have

$$\langle dz\, \partial\phi(z)\ \ dw\, \partial\phi(w)\rangle = -\frac{dz \wedge dw}{(z-w)^2} \tag{1}$$

where we can interpret the left side as the correlator of two copies of the holomorphic Dolbeaux differential of ϕ placed at z and w. Prove that the right hand side of (1) is a $PSL_2(\mathbb{C})$-*invariant* 2-form on the configuration space of two non-coinciding points on $\mathbb{C}$.[1] Is this form *horizontal* w.r.t. $PSL_2(\mathbb{C})$-action on the configuration space?

2. "Toy Sugawara construction" for the free boson

Recall that in the free boson theory, the stress-energy tensor is given by

$$\hat{T}(z) =: -\frac{1}{2}\partial\hat{\phi}(z)\partial\hat{\phi}(z) : \tag{2}$$

and in terms of creation-annihilation operators $\hat{a}_n$ satisfying

$$[\hat{a}_n, \hat{a}_m] = n\delta_{n,-m}\mathbf{1} \tag{3}$$

one has

$$i\partial\hat{\phi}(z) = \sum_{n=-\infty}^{\infty} \hat{a}_n z^{-n-1}$$

Also, recall that the normal ordering $:\cdots:$ puts annihilation operators $\hat{a}_{\geq 0}$ to the right and creation operators $\hat{a}_{<0}$ to the left.

(a) Using (2) and the definition of Virasoro generators via stress-energy tensor,

$$\hat{L}_n = \oint \frac{dz}{2\pi i} z^{n+1}\hat{T}(z) \tag{4}$$

(with the simple closed integration contour going counterclockwise around the origin), show that one has the following expression for $\hat{L}_n$ in terms of creation/annihilation operators:

$$\hat{L}_n = \frac{1}{2}\sum_{k=-\infty}^{\infty} : \hat{a}_k\hat{a}_{n-k} : \tag{5}$$

[1]Hint. One possible way: check compatibility with translations, scalings, rotations and the map $z \mapsto 1/z$ (and show that this is sufficient for Möbius invariance). Second possibility: express the r.h.s. of (1) as a derivative of the logarithm of a cross-ratio between z, w and two arbitrary fixed points x, y in z and w; then use the Möbius-invariance of the cross-ratio. Third possibility: invent your very own way!

(b) Prove directly from Heisenberg commutation relations (3) that generators (5) satisfy Virasoro relations

$$[\hat{L}_n, \hat{L}_m] = (n-m)\hat{L}_{n+m} + c\frac{n^3-n}{12}\delta_{n,-m}\mathbf{1} \tag{6}$$

with central charge $c = 1$.

3. From TT OPE to Virasoro relations

For a general CFT on a plane, one has the operator product expansion of the stress-energy tensor with itself of the form

$$\mathcal{R}\hat{T}(z)\hat{T}(w) = \frac{c/2}{(z-w)^4}\mathbf{1} + \frac{2\hat{T}(w)}{(z-w)^2} + \frac{\partial\hat{T}(w)}{z-w} + \text{reg} \tag{7}$$

with c some number (the central charge of the theory). Prove that (7) implies Virasoro commutation relations (6) for the generators $\hat{L}_n$ defined as Fourier modes of $\hat{T}$, via (4).

4. Field-state correspondence

The idea of associating a state to a field Φ is to let the field operator $\hat{\Phi}(z,\bar{z})$ act on the vacuum and take a limit $z \to 0$.

(a) For the free boson theory, show that limiting state

$$\lim_{z\to 0} \partial^n \hat{\phi}(z)|\text{vac}\rangle$$

is well-defined for $n \geq 1$. Express it in terms of the standard basis for the space of states.

(b) Same question about

$$\lim_{z\to 0} \hat{T}(z)|\text{vac}\rangle$$

5. Some correlators

In the free boson theory, use Wick's lemma to find the following correlators.

(a) $\langle : \partial\phi(z)^3 : \ : \partial\phi(w)^3 : \rangle = \Big\langle \text{vac} \Big| \mathcal{R} : \partial\hat{\phi}(z)^3 : \ : \partial\hat{\phi}(w)^3 : \Big| \text{vac} \Big\rangle$.

(b) $\langle T(z_1)\, T(z_2)\, T(z_3) \rangle = \Big\langle \text{vac} \Big| \mathcal{R}\ \hat{T}(z_1)\, \hat{T}(z_2)\, \hat{T}(z_3) \Big| \text{vac} \Big\rangle$.

Here $\mathcal{R}$ is the radial ordering.