1/15/2020

1.1. Systems of linear equations

(Lay, Lay, McDonald)

Linear equation:

$$\underbrace{a_1}_{\text{coefficients}} \overbrace{x_1}^{\text{variables (indeterminates)}} + a_2 x_2 + \dots + a_n x_n = b$$

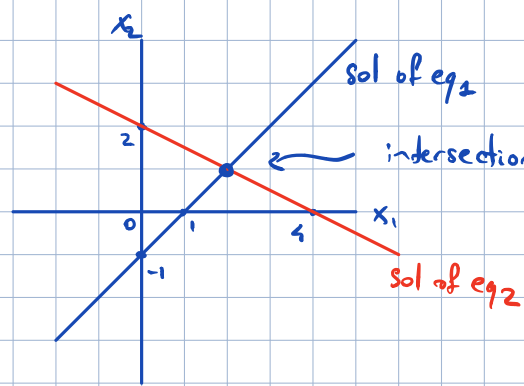
coefficients ← given real/complex numbers

System of linear equations:

m equations on n variables ⟶ set of all solutions ("solution set")

<u>Ex</u>: (a) $x_1 - x_2 = 1$ (eq_1)

$x_1 + 2x_2 = 4$ (eq_2)



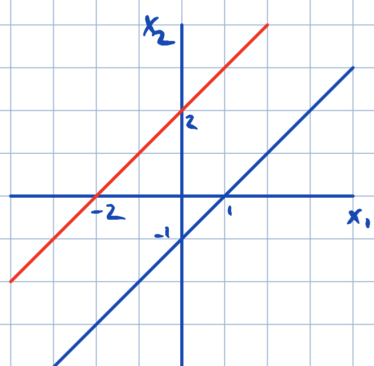
intersection - solution of the system

$(x_1, x_2) = (2, 1)$

- <u>unique</u> solution

(b) $x_1 - x_2 = 1$

$-2x_1 + 2x_2 = 4$



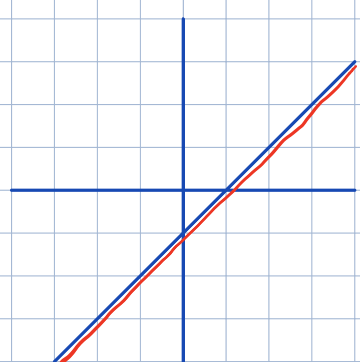
parallel lines

→ <u>no solutions</u>

⇒ system is <u>inconsistent</u>

(c) $x_1 - x_2 = 1$

$-2x_1 + 2x_2 = -2$



lines coincide

→ <u>infinitely many solutions</u>

$x_1 = x_2 + 1$, x_2 any number

Any system of lin. eq. has either or

① no solutions] system inconsistent

② exactly one solution

③ infinitely many solutions] system consistent

Matrix notation

Linear system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 3x_2 - 3x_3 &= 6 \\ 2x_1 \qquad + 3x_3 &= 3 \end{aligned}$$

(coefficients of each variable aligned in columns)

→ matrix of coefficients $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 3 & -3 \\ 2 & 0 & 3 \end{bmatrix}$

3 rows (3 equations)
3 columns (3 variables) } ⇒ matrix of size 3×3

augmented matrix $\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 3 & -3 & 6 \\ 2 & 0 & 3 & 3 \end{array}\right]$ a 3×4 matrix

added a column of right-hand sides

Solving a linear system

idea use x_1 - term in eq_1 to eliminate x_1 from the other equations
use x_2 - term in eq_2 to eliminate x_2 from the other eq. etc.
⇝ obtain a very simple equivalent linear sys. (i.e. with the same solution set)

Ex:
$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 3x_2 - 3x_3 &= 6 \\ 2x_1 \qquad + 3x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 3 & -3 & 6 \\ 2 & 0 & 3 & 3 \end{bmatrix}$$

- keep x_1 in eq_1 and eliminate it from other eq.: add $(-2)\cdot eq_1$ to eq_3

$r_3 \mapsto r_3 - 2r_1$

$$\begin{array}{l|r} -2\cdot eq_1 & -2x_1 + 4x_2 - 2x_3 = 0 \\ + eq_3 & 2x_1 \qquad + 3x_3 = 3 \\ \hline \text{new } eq_3 & 4x_2 + x_3 = 3 \end{array}$$

new system: $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 3 & -3 & 6 \\ 0 & 4 & 1 & 3 \end{bmatrix}$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 3x_2 - 3x_3 &= 6 \\ 4x_2 + x_3 &= 3 \end{aligned}$$

- multiply eq_2 by $\frac{1}{3}$, to get 1 as the coeff of x_2 in eq_2 (optional - simplifies the next step)

$r_2 \mapsto r_2 \cdot \frac{1}{3}$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - x_3 &= 2 \\ 4x_2 + x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 4 & 1 & 3 \end{bmatrix}$$

- use x_2-term in eq_2 to eliminate x_2 from eq_3 : replace eq_3 with $eq_3 - 4 \cdot eq_2$

$$\begin{array}{rl} -4 \cdot eq_2 & -4x_2 + 4x_3 = -8 \\ eq_3 & 4x_2 + x_3 = 3 \\ \hline \text{new } eq_3 & 5x_3 = -5 \end{array}$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - x_3 &= 2 \\ 5x_3 &= -5 \end{aligned}$$

$r_3 \mapsto r_3 - 4r_2$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 5 & -5 \end{bmatrix}$$

- multiply eq_3 by $\frac{1}{5}$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - x_3 &= 2 \\ x_3 &= -1 \end{aligned}$$

$r_3 \mapsto r_3 \cdot \frac{1}{5}$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

a system in "triangular" (or "echelon") form

- Eliminate x_3 from eq_1, eq_2 : $eq_2 \mapsto eq_2 + eq_3$, $eq_1 \mapsto eq_1 - eq_3$

$r_2 \mapsto r_2 + r_3$
$r_1 \mapsto r_1 - r_3$

$$\begin{aligned} x_1 - 2x_2 &= 1 \\ x_2 &= 1 \\ x_3 &= -1 \end{aligned}$$

$$\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

- Eliminate x_2 from eq_1 : $eq_1 \to eq_1 + 2eq_2$

$r_1 \mapsto r_1 + 2r_2$

$$\begin{aligned} x_1 &= 3 \\ x_2 &= 1 \\ x_3 &= -1 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$\hookrightarrow$ we proved that the only solution of the original sys. is $(3, 1, -1)$

check (substitute into orig. sys.)

$$\begin{aligned} 3 - 2\cdot 1 + (-1) &= 0 \\ 3 \cdot 1 - 3(-1) &= 6 \\ 2 \cdot 3 + 3(-1) &= 3 \end{aligned}$$ ✓

Solving a lin. sys., we use the operations:

1. replace an eq. with itself plus a multiple of another eq.
2. interchange two equations
3. multiply all terms in an equation by a nonzero constant

For the augmented matrix, we perform the corresponding elementary row operations

1. (replacement) replace a row with itself plus a multiple of another row
2. (interchange) interchange two rows
3. (scaling) multiply all entries in a row by a nonzero constant

def Two matrices are **row equivalent** iff they can be transformed one into another by a sequence of elem. row operations

- Row operations are reversible.
- If the augm. matrices of two lin. sys. are row equivalent, then the two systems have the same solution set.

1.2. Row reduction and echelon forms

- **leading entry** in a row = leftmost nonvanishing entry
- a rectangular matrix is in **row echelon form** (REF) if
  - all nonzero rows are above zero rows
  - each leading entry in a row is (in a column) to the right of the leading entry in a row above it
  - all entries in a column below a leading entry are zero.

pivot columns

pivot positions

Ex: $\begin{bmatrix} \bullet & * & * & * \\ 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & \bullet & * & * & * & * & * \\ 0 & 0 & 0 & \bullet & * & * & * \\ 0 & 0 & 0 & 0 & \bullet & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \bullet \end{bmatrix}$

$\underset{\neq 0}{\bullet}$ - leading entries $*$ - any entries

A matrix is in **reduced row echelon form** (RREF) if additionally
- all leading entries are 1
- each leading 1 is the only nonzero entry in its column.

Ex: $\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & * & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 1 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 1 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

- Any matrix can be row reduced (transformed by a sequence of elem. row op.) into **more than one** matrix in **REF**. However, **RREF of a matrix is unique.**
- Leading entries are always in same positions for any REF of A = **pivot positions**. A column containing a pivot position = "**pivot column**"

Leading entries in a REF of A = **pivots.**
(numbers)

Row reduction algorithm

$$\text{matrix } A \xrightarrow[\text{"forward phase"}]{\text{Steps I–IV}} \text{REF of } A \xrightarrow[\text{"backward phase"}]{\text{step V}} \text{RREF of } A$$

Ex:

$$A = \begin{bmatrix} 0 & 2 & -6 & -1 & -2 \\ 2 & 1 & 9 & 9 & 6 \\ 2 & 4 & 0 & 6 & 0 \end{bmatrix}$$

Step I: begin with leftmost nonzero column. It is a pivot column; pivot pos. is at the top

Step II: select a nonzero entry in pivot col. as pivot. If necessary, interchange rows to move this entry into pivot pos.

$$\xrightarrow[r_1 \rightleftarrows r_3]{\text{interchange}} \begin{bmatrix} 2 & 4 & 0 & 6 & 0 \\ 2 & 1 & 9 & 9 & 6 \\ 0 & 2 & -6 & -1 & -2 \end{bmatrix}$$

Step III Use row replacement to create zeros in all positions below the pivot

$$\xrightarrow[r_2 \mapsto r_2 - r_1]{} \begin{bmatrix} 2 & 4 & 0 & 6 & 0 \\ 0 & -3 & 9 & 3 & 6 \\ 0 & 2 & -6 & -1 & -2 \end{bmatrix}$$

Step IV Cover (or ignore) the row containing pivot pos. and all rows above it.

Apply steps I – III to the remaining submatrix.

Repeat until there are no nonzero rows to modify.

$$\begin{bmatrix} 2 & 4 & 0 & 6 & 0 \\ 0 & -3 & 9 & 3 & 6 \\ 0 & 2 & -6 & -1 & -2 \end{bmatrix} \xrightarrow[r_3 \to r_3 + \frac{2}{3} r_2]{} \begin{bmatrix} 2 & 4 & 0 & 6 & 0 \\ 0 & -3 & 9 & 3 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[r_2 \to -\frac{1}{3} r_2]{\text{(optional)}} \begin{bmatrix} 2 & 4 & 0 & 6 & 0 \\ 0 & 1 & -3 & -1 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

new pivot

new pivot

REF!

already in REF.
$\Rightarrow$ IV stops

If we want RREF:

Step V: beginning with rightmost pivot and working upward and to the left, create zeros above each pivot. If pivot is not 1, make it 1 by rescaling rows

$$\begin{bmatrix} 2 & 4 & 0 & 6 & 0 \\ 0 & 1 & -3 & -1 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[r_1 \mapsto r_1 - 6r_3]{r_2 \mapsto r_2 + r_3} \begin{bmatrix} 2 & 4 & 0 & 0 & -12 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_1 \to r_1 - 4r_2} \begin{bmatrix} 2 & 0 & 12 & 0 & -12 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

created zeros above pivots

$$\xrightarrow[r_1 \to r_1 \cdot \frac{1}{2}]{\text{rescale}} \begin{bmatrix} 1 & 0 & 6 & 0 & -6 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \longleftarrow \text{RREF of } A$$

Solutions of lin. sys.

Suppose augm. mat. of a lin. sys. has been reduced to RREF

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

I.e. system
$$\begin{aligned} x_1 + \quad 3x_3 &= -1 \\ x_2 + 2x_3 &= 5 \\ 0 &= 0 \end{aligned}$$

variables x_1, x_2 corresponding to pivot columns are "basic variables"; var. x_3 corresp to a non-pivot col. is a "free variable".

Can solve for basic variables in terms of free variables:

$$\begin{cases} x_1 = -1 - 3x_3 \\ x_2 = 5 - 2x_3 \end{cases}$$
x_3 is free (takes any value)

– description of <u>all</u> sol.s of the lin. sys.

e.g. can take $x_3 = 1 \rightsquigarrow (-4, 3, 1)$ is a sol.

$-4 = -1-3\cdot 1$, $3 = 5-2\cdot 1$

- A system is <u>consistent</u> iff REF of the augm. mat. does not have a row of form $[0 \,..\, 0 \;\; b]$, $b \neq 0$ $\Leftrightarrow$ $0 = b$ contradictory eq.
(i.e. iff the last column is not pivotal)

- Solution of a consistent sys. is <u>unique</u> iff there are no free variables, i.e. no non-pivot columns (except the last one)