1/17/2020

## Row reduction algorithm

$$\text{matrix } A \xrightarrow[\text{"forward phase"}]{\text{steps I–IV}} \text{REF of } A \xrightarrow[\text{"backward phase"}]{\text{step V}} \text{RREF of } A$$

LAST TIME

**Ex:**

$$A = \begin{bmatrix} 0 & 2 & -6 & -1 & -2 \\ 2 & 1 & 9 & 9 & 6 \\ 2 & 4 & 0 & 6 & 0 \end{bmatrix}$$

**Step I:** begin with leftmost nonzero column. It is a pivot column; pivot pos. is at the top

**Step II:** select a nonzero entry in pivot col. as pivot. If necessary, interchange rows to move this entry into pivot pos.

$$\xrightarrow[r_1 \rightleftarrows r_3]{\text{interchange}} \begin{bmatrix} 2 & 4 & 0 & 6 & 0 \\ 2 & 1 & 9 & 9 & 6 \\ 0 & 2 & -6 & -1 & -2 \end{bmatrix}$$

**Step III** Use row replacement to create zeros in all positions below the pivot

$$\xrightarrow[r_2 \mapsto r_2 - r_1]{} \begin{bmatrix} 2 & 4 & 0 & 6 & 0 \\ 0 & -3 & 9 & 3 & 6 \\ 0 & 2 & -6 & -1 & -2 \end{bmatrix}$$

**Step IV** Cover (or ignore) the row containing pivot pos. and all rows above it.

Apply steps I–III to the remaining submatrix.

Repeat until there are no nonzero rows to modify.

$$\begin{bmatrix} 2 & 4 & 0 & 6 & 0 \\ 0 & -3 & 9 & 3 & 6 \\ 0 & 2 & -6 & -1 & -2 \end{bmatrix} \xrightarrow[r_3 \to r_3 + \frac{2}{3} r_2]{} \begin{bmatrix} 2 & 4 & 0 & 6 & 0 \\ 0 & -3 & 9 & 3 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[r_2 \to -\frac{1}{3} r_2]{\text{(optional)}} \begin{bmatrix} 2 & 4 & 0 & 6 & 0 \\ 0 & 1 & -3 & -1 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

new pivot

new pivot

already in REF.
⇒ IV stops

REF!

### If we want RREF:

**Step V:** beginning with rightmost pivot and working upward and to the left, create zeros above each pivot. If pivot is not 1, make it 1 by rescaling rows

$$\begin{bmatrix} 2 & 4 & 0 & 6 & 0 \\ 0 & 1 & -3 & -1 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow[r_1 \mapsto r_1 - 6r_3]{r_2 \mapsto r_2 + r_3} \begin{bmatrix} 2 & 4 & 0 & 0 & -12 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_1 \mapsto r_1 - 4r_2} \begin{bmatrix} 2 & 0 & 12 & 0 & -12 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

created zeros above pivots

$$\xrightarrow[r_1 \mapsto r_1 \cdot \frac{1}{2}]{\text{rescale}} \begin{bmatrix} 1 & 0 & 6 & 0 & -6 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \longleftarrow \text{RREF of } A$$

<u>Solutions of lin. sys.</u> Suppose augm. mat. of a lin. sys. has been reduced to RREF

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

I.e. system
$$\begin{aligned} x_1 \quad + 3x_3 &= -1 \\ x_2 + 2x_3 &= 5 \\ 0 &= 0 \end{aligned}$$

variables $x_1, x_2$ corresponding to pivot columns are "basic variables"; var. $x_3$ corresp to a non-pivot col. is a "free variable".

Can solve for basic variables in terms of free variables:

$$\begin{cases} x_1 = -1 - 3x_3 \\ x_2 = 5 - 2x_3 \\ x_3 \text{ is free (takes any value)} \end{cases}$$

– description of <u>all</u> sols of the lin. sys.

e.g. can take $x_3 = 1 \rightsquigarrow (\underset{-1-3\cdot 1}{-4}, \underset{5-2\cdot 1}{3}, 1)$ is a sol.

- A system is <u>consistent</u> iff REF of the augm. mat. does not have a row of form $[0 \, .. \, 0 \;\; b]$, $b \neq 0$ $\iff$ $0 = b$ contradictory eq. (i.e. iff the last column is not pivotal)
- Solution of a consistent sys. is <u>unique</u> iff there are no free variables, i.e. no non-pivot columns (except the last one)

# 1.3. Vector equations

- vector = ordered list of numbers
- column vector = matrix with only one column

## Vectors in $\mathbb{R}^2$

Ex: $\begin{bmatrix} 1.3 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ Set of vectors with two (real) entries $=: \mathbb{R}^2$

- vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^2$ are equal if their corresponding components are equal.
  E.g. $\begin{bmatrix} 2 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 3 \\ 2 \end{bmatrix}$
- for $\vec{u}, \vec{v} \in \mathbb{R}^2$, can form the <u>sum</u> $\vec{u} + \vec{v}$ by adding corresp. entries of $u, v$
  e.g. $\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 2+(-1) \\ 3+5 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$
- for $\vec{u} \in \mathbb{R}^2$, $c$ a number, the scalar multiple $c \cdot \vec{u}$ is obtained by multiplying each entry of $u$ by $c$:

$$3 \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \cdot 2 \\ 3 \cdot 7 \end{bmatrix} = \begin{bmatrix} 6 \\ 21 \end{bmatrix}$$

Ex: for $\vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, find $5\vec{u}$, $-2\vec{v}$, $5\vec{u} + (-2\vec{v})$

Sol: $5\vec{u} = \begin{bmatrix} 10 \\ -5 \end{bmatrix}$ $-2\vec{v} = \begin{bmatrix} -6 \\ -4 \end{bmatrix}$ $5\vec{u} + (-2\vec{v}) = \begin{bmatrix} 4 \\ -9 \end{bmatrix}$

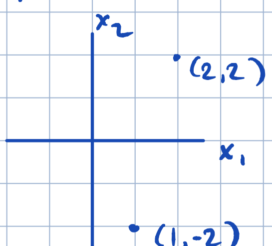
**Notation:** can write $\underset{\substack{2 \times 1 \\ \text{matrix}}}{\begin{bmatrix} 2 \\ 7 \end{bmatrix}} = (2, 7) \neq \underset{\substack{1 \times 2 \\ \text{matrix}}}{\begin{bmatrix} 2 & 7 \end{bmatrix}}$

comma! - shorthand for a column vector

## Geometric description of vectors in $\mathbb{R}^2$

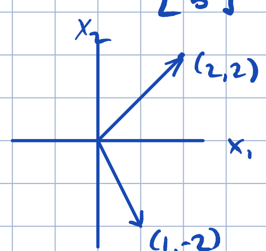
point $(a, b)$ on the coordinate plane $\sim \begin{bmatrix} a \\ b \end{bmatrix}$

So: $\mathbb{R}^2$ = set of all points on the plane



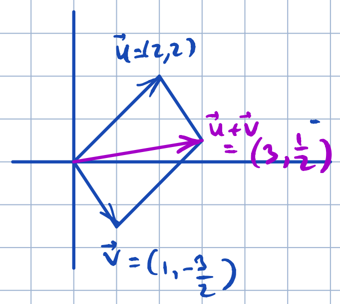
vectors as points

or



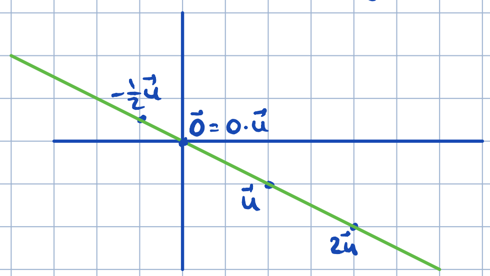
vectors as arrows

**Sum:** parallelogram rule




- 4-th vertex of the parallelogram w/ vertices at $0, \vec{u}, \vec{v}$

**multiples** Ex: $\vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ Display $\vec{u}, 2\vec{u}, -\frac{1}{2}\vec{u}$




## Vectors in $\mathbb{R}^n$

$\mathbb{R}^n$ = set of vectors of form $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ ($n \times 1$ matrices)

- can add two vectors in $\mathbb{R}^n$ $\vec{u}+\vec{v}$. (must be of the same size!)
- can form a scaler multiple $c\cdot\vec{u}$

(one has commutativity, associativity, distributivity like for numbers)

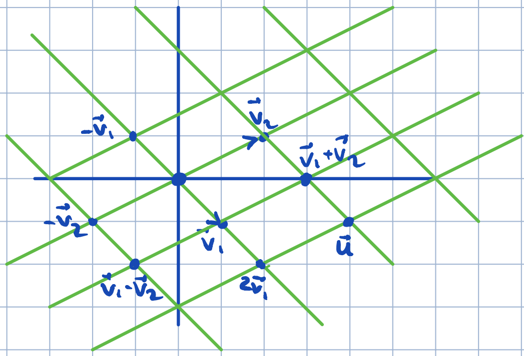
- zero vector $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\vec{0}+\vec{u} = \vec{u}$
- negative of a vector $-\vec{u} = (-1)\cdot\vec{u} = \begin{bmatrix} -u_1 \\ -u_2 \\ \vdots \\ -u_n \end{bmatrix}$ Then $(-\vec{u})+\vec{u} = \vec{0}$
- Also, notation: $\vec{u}+(-\vec{v}) =: \vec{u}-\vec{v}$

## Linear combinations

For vectors $\vec{v}_1, \dots, \vec{v}_p \in \mathbb{R}^n$ and scalars $c_1, \dots, c_p$, the vector $\vec{y} = c_1\vec{v}_1 + \dots + c_p\vec{v}_p$ is called the <u>linear combination</u> of $\vec{v}_1, \dots, \vec{v}_p$ with weights $c_1, \dots, c_p$

**Ex:** some lin. comb. of $\vec{v}_1, \vec{v}_2$ : $3\vec{v}_1 - \frac{5}{7}\vec{v}_2$, $\frac{1}{2}\vec{v}_1 = \frac{1}{2}\vec{v}_1 + 0\cdot\vec{v}_2$, $\vec{0} = 0\cdot\vec{v}_1 + 0\cdot\vec{v}_2$

Ex: Some lin. comb. of $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$:




Q: express $\vec{u}$ as a lin. comb. of $\vec{v}_1$ and $\vec{v}_2$

Sol: by parallelogram rule, $\vec{u} = 2\vec{v}_1 + \vec{v}_2$

Ex: $\vec{a}_1 = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -1 \\ -1 \\ -4 \end{bmatrix}$

Q: can $\vec{b}$ be generated as a lin. comb. of $\vec{a}_1, \vec{a}_2$?

I.e. can we find weights $x_1, x_2$ such that $\boxed{x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b}}$ ? (*)

(*) means

$$x_1 \underbrace{\begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}}_{\vec{a}_1} + x_2 \underbrace{\begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}}_{\vec{a}_2} = \underbrace{\begin{bmatrix} -1 \\ -1 \\ -4 \end{bmatrix}}_{\vec{b}}$$

$$\text{l.h.s.} = \begin{bmatrix} x_1 \\ -3x_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ -5x_2 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 \\ -3x_1 - 5x_2 \\ -x_1 + 2x_2 \end{bmatrix}$$

So: (*) is true iff $\begin{cases} x_1 + 3x_2 = -1 \\ -3x_1 - 5x_2 = -1 \\ -x_1 + 2x_2 = -4 \end{cases}$ - linear system!

Augmented matrix:

$$\begin{bmatrix} 1 & 3 & -1 \\ -3 & -5 & -1 \\ -1 & 2 & -4 \end{bmatrix} \underset{\text{row equiv}}{\sim} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 4 & -4 \\ 0 & 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \\ 0 & 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

RREF

So, the system is equiv. to

$$x_1 = 2$$
$$x_2 = -1$$
$$0 = 0$$

thus: $\boxed{2\vec{a}_1 + (-1)\vec{a}_2 = \vec{b}}$ - $\vec{b}$ is a lin. comb. of $\vec{a}_1, \vec{a}_2$

$$2\begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix} + (-1)\begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -4 \end{bmatrix}$$

Note: the augm. mat. of our system was

$$\begin{array}{ccc} \begin{bmatrix} 1 & 3 & -1 \\ -3 & -5 & -1 \\ -1 & 2 & -4 \end{bmatrix} \\ \vec{a}_1 \quad \vec{a}_2 \quad \vec{b} \end{array} \overset{\text{notation}}{=} [\vec{a}_1 \ \vec{a}_2 \ \vec{b}]$$

- Equation $x_1 \vec{a}_1 + \dots + x_p \vec{a}_p = \vec{b}$ has the same solution set as the lin. sys. whose augm. mat. is $[\vec{a}_1 \dots \vec{a}_p \ \vec{b}]$ (**)

In particular, $\vec{b}$ can be generated as a lin. comb. of $\vec{a}_1, \dots, \vec{a}_p$ iff there exists a solution for the lin. sys. corresponding to the matrix (**)

Def Let $\vec{v}_1, \dots, \vec{v}_p \in \mathbb{R}^n$. The set of all lin. comb. of $\vec{v}_1, \dots, \vec{v}_p$ is denoted

Span$\{\vec{v}_1, \dots, \vec{v}_p\}$ = "subset of $\mathbb{R}^n$ spanned (or generated) by $\vec{v}_1, \dots, \vec{v}_p$"

I.e. Span$\{\vec{v}_1, \dots, \vec{v}_p\}$ = set of all vectors that can be written in the form $c_1 \vec{v}_1 + \dots + c_p \vec{v}_p$ with $c_1, \dots, c_p$ scalars

- vector $\vec{b}$ is in Span$\{\vec{v}_1, \dots, \vec{v}_p\}$ iff vector equation $x_1 \vec{v}_1 + \dots + x_p \vec{v}_p = \vec{b}$ has a solution <=> lin. sys. with Augm. Mat. $[\vec{v}_1 \dots \vec{v}_p \ \vec{b}]$ has a solution

- Span$\{\vec{v}_1, \dots, \vec{v}_p\}$ contains every scalar multiple of $\vec{v}_1$ and in particular contains $\vec{0}$.