1/22/2020

LAST TIME: $\vec{a}_1 = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -1 \\ -1 \\ -4 \end{bmatrix}$

Q: can we write $\vec{b} = x_1\vec{a}_1 + x_2\vec{a}_2$ (*) for some x_1, x_2?

Sol: (*) ~ lin. sys. with Aug. Mat.
$$\begin{array}{ccc} \vec{a}_1 & \vec{a}_2 & \vec{b} \end{array}$$
$$\begin{bmatrix} -1 & 3 & -1 \\ -3 & -5 & -1 \\ -1 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \boxed{\begin{matrix} x_1 = 2 \\ x_2 = -1 \end{matrix}} \quad 0 = 0$$
$= [\vec{a}_1\ \vec{a}_2\ \vec{b}]$ (Notation)

- Equation $x_1\vec{a}_1 + \dots + x_p\vec{a}_p = \vec{b}$ has same sol. set as lin. sys. with Aug. Mat. $A = \boxed{[\vec{a}_1 \cdots \vec{a}_n\ \vec{b}]}$
  (in particular, $\vec{b}$ can be generated as a lin. comb. of $\vec{a}_1, \dots, \vec{a}_p$, if the lin. sys. corresp. to A has a solution)

__def.__ Let $\vec{v}_1, \dots, \vec{v}_p \in \mathbb{R}^n$. The set of all linear combinations of $\vec{v}_1, \dots, \vec{v}_p$ is denoted

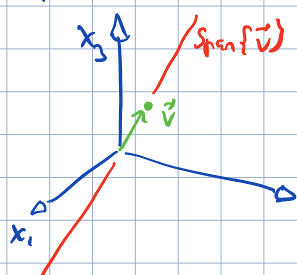
$$\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\} = \{c_1\vec{v}_1 + \dots + c_p\vec{v}_p \mid c_1, \dots, c_p \in \mathbb{R}\}$$
= the subset of $\mathbb{R}^n$ __spanned__ (or __generated__) by $\vec{v}_1, \dots, \vec{v}_p$.

- a vector $\vec{b}$ is in $\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$ iff the vector eq. $x_1\vec{v}_1 + \dots + x_p\vec{v}_p = \vec{b}$ has a sol.
  $\Leftrightarrow$ lin. sys. with Aug. Mat. $[\vec{v}_1 \dots \vec{v}_p\ \vec{b}]$ has a sol.

- $\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$ contains every scalar multiple of $\vec{v}_1$ and contains $\vec{0}$.

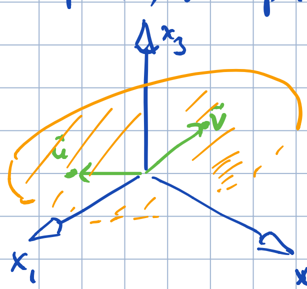
__Geom. description of $\text{Span}\{\vec{v}\}$, $\text{Span}\{\vec{u}, \vec{v}\}$__

- Let $\vec{v} \in \mathbb{R}^3$ a non zero vector. $\text{Span}\{\vec{v}\} = \{$scalar multiples of $\vec{v}\}$
  = pts on the line in $\mathbb{R}^3$ through $\vec{0}$ and $\vec{v}$



- Let $\vec{u}, \vec{v} \in \mathbb{R}^3$ two non-zero vectors s.t. $\vec{v}$ is not a multiple of $\vec{u}$.

$\text{Span}\{\vec{u}, \vec{v}\}$ = plane in $\mathbb{R}^3$ through $\vec{0}, \vec{u}, \vec{v}$.



Ex: Q: $\vec{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\vec{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$ Is $\vec{b}$ in the plane Span$\{\vec{a}_1, \vec{a}_2\}$?

Sol: does $x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b}$ have a sol?

$$\begin{array}{c} \begin{bmatrix} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{bmatrix} \\ \begin{array}{ccc} \vec{a}_1 & \vec{a}_2 & \vec{b} \end{array} \end{array} \sim \underbrace{\begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & -2 \end{bmatrix}}_{\text{REF}}$$

$0 = -2 \Rightarrow$ no solutions!

$\Rightarrow \vec{b} \notin$ Span !

1.4. Matrix Equation $A\vec{x} = \vec{b}$

Def for an $m \times n$ matrix $A = [\vec{a}_1 \cdots \vec{a}_n]$ (columns) and $\vec{x} \in \mathbb{R}^n$, the matrix-vector product is

$$A\vec{x} = [\vec{a}_1 \cdots \vec{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} := x_1\vec{a}_1 + \cdots + x_n\vec{a}_n$$

- lin. comb. of col. of A with weights given by entries of $\vec{x}$.

Note: $A\vec{x}$ is only defined if # columns in A = # entries in $\vec{x}$.

Ex:

$$\begin{array}{c} \begin{bmatrix} 1 & -2 & 3 \\ -4 & 5 & 0 \end{bmatrix} \\ \begin{array}{ccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{array} \end{array} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -4 \end{bmatrix} + (-1)\begin{bmatrix} -2 \\ 5 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -8 \end{bmatrix} + \begin{bmatrix} 2 \\ -5 \end{bmatrix} + \begin{bmatrix} 9 \\ 0 \end{bmatrix} = \begin{bmatrix} 13 \\ -13 \end{bmatrix}$$

Q for $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^m$, write lin. comb. $\vec{u} = 2\vec{v}_1 - 5\vec{v}_2 + 3\vec{v}_3$ as a matrix vector product

Sol: $$\vec{u} = \underbrace{[\vec{v}_1 \;\; \vec{v}_2 \;\; \vec{v}_3]}_{m\times 3 \text{ matrix}} \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

Lin. sys $\begin{matrix} x_1 - 2x_2 + 3x_3 = 5 \\ -4x_1 + 5x_2 = 7 \end{matrix}$ $\Leftrightarrow$ vector eq. $x_1\begin{bmatrix} 1 \\ -4 \end{bmatrix} + x_2\begin{bmatrix} -2 \\ 5 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ $\Leftrightarrow$ $\begin{bmatrix} 1 & -2 & 3 \\ -4 & 5 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

"matrix eq." of form $A\vec{x} = \vec{b}$

matrix of coeff of the lin. sys.

- If $A = [\vec{a}_1 \cdots \vec{a}_n]$ $m \times n$ matrix, $\vec{b} \in \mathbb{R}^m$, then matrix eq. $A\vec{x} = \vec{b}$ has same sol. set as the vect. eq. $x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{b}$ and same as lin. sys. with Aug. Mat. $[\vec{a}_1 \cdots \vec{a}_n \; \vec{b}]$

Existence of solutions

- $A\vec{x} = \vec{b}$ has a solution iff $\vec{b}$ is a lin. comb. of columns of A ($\Leftrightarrow \vec{b} \in$ Span$\{\vec{a}_1, \ldots, \vec{a}_n\}$)

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ Q: Is eq. $A\vec{x} = \vec{b}$ consistent for all $\vec{b}$?

Sol: Aug. Mat $= \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & -6 & -12 & b_3 - 7b_1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{bmatrix}$

want this to vanish for consistency

$\Rightarrow$ $A\vec{x}=\vec{b}$ consistent iff $b_1 - 2b_2 + b_3 = 0$.

- eq. of a plane through the origin in $\mathbb{R}^3$
$= \text{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$

- $A\vec{x}=\vec{b}$ is not consistent for every $\vec{b}$, because REF of A has a row of zeros

  If A had a pivot in each row, REF $[A\ \vec{b}]$ would be of form $\left[\begin{array}{ccc|c} \bullet & * & * & * \\ 0 & \bullet & * & * \\ 0 & 0 & \bullet & * \end{array}\right]$

  - consistent $\forall \vec{b}$.

Theorem Let A be an $m\times n$ matrix. The following are equivalent.

(a) for each $\vec{b} \in \mathbb{R}^m$, eq. $A\vec{x}=\vec{b}$ has a solution

(b) each $\vec{b} \in \mathbb{R}^m$ is a lin. comb. of columns of A

(c) columns of A span entire $\mathbb{R}^m$

$\boxed{(d)}$ A has a pivot in every row

note: coeff. matrix, not the augm. mat.

Row-vector rule for computing $A\vec{x}$

If $A\vec{x}$ is defined, then i^{th} entry of $A\vec{x}$ is the sum of products of corresponding entries from row i of A and from $\vec{x}$.

Ex: $$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + x_3\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 4x_1 + 5x_2 + 6x_3 \\ 7x_1 + 8x_2 + 9x_3 \end{bmatrix}$$

"dot product"
$$\begin{bmatrix} 1 & 2 & 3 \\ \hline & & \\ \hline \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \begin{bmatrix} \hline 4 & 5 & 6 \\ \hline \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Ex: $$\begin{bmatrix} 1 & -2 & 3 \\ -4 & 5 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1\cdot 2 + (-2)(-1) + 3\cdot 3 \\ (-4)\cdot 2 + 5\cdot(-1) + 0\cdot 3 \end{bmatrix} = \begin{bmatrix} 13 \\ -13 \end{bmatrix}$$

- **Properties of matrix-vector product $A\vec{x}$**

  for A an $m\times n$ matrix, $\vec{u}, \vec{v} \in \mathbb{R}^n$ and c a scalar:

  (a) $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$

  (b) $A(c\vec{u}) = c(A\vec{u})$