1.7. Linear independence

def A set of vectors $\{\vec{v}_1, \dots, \vec{v}_p\}$ in $\mathbb{R}^n$ is <u>linearly independent</u> if the vector eq. $\boxed{x_1\vec{v}_1 + \dots + x_p\vec{v}_p = \vec{0}}$ has only the trivial solution.

Set $\{\vec{v}_1, \dots, \vec{v}_p\}$ is <u>linearly dependent</u> if there exist weights $c_1, \dots, c_p$ (not all zero), s.t. $\boxed{c_1\vec{v}_1 + \dots + c_p\vec{v}_p = \vec{0}}$ – "linear dependence relation among $\vec{v}_1, \dots, \vec{v}_p$"

Ex: $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} -1 \\ 6 \\ 0 \end{bmatrix}$

Q: (1) Is the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ lin. dep. ?
(2) If yes, find a linear dependence relation.

Sol:

Vec. eq. $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$ (*)

Aug. mat.
$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 0 & 6 & 0 \\ 2 & 3 & 0 & 0 \end{bmatrix} \sim \underset{\text{REF}}{\begin{array}{c} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -4 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \end{array}} \sim \underset{\text{RREF}}{\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

free var. (x_3)

(1) x_3 is free ⇒ (*) has (infinitely many) nontriv. solutions

(2) from RREF:
$x_1 = -3x_3$
$x_2 = 2x_3$
x_3 free

e.g. for $x_3 = -4$, we have a sol. $(12, -8, -4)$

⇒ lin. dep. rel. $\boxed{12\vec{v}_1 - 8\vec{v}_2 - 4\vec{v}_3 = \vec{0}}$

(one of infinitely many lin. dep. rel.)

Lin. independence of matrix columns

for $A = [\vec{a}_1 \cdots \vec{a}_n]$ a matrix, eq. $A\vec{x} = \vec{0}$ <=> $x_1\vec{a}_1 + \dots + x_n\vec{a}_n = \vec{0}$

So:
- each lin. dep. rel. among columns of A corresponds to a sol. of $A\vec{x} = \vec{0}$
- columns of A are lin. indep. iff $A\vec{x} = \vec{0}$ has <u>only</u> the triv. sol.

Ex: $A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$ are colums lin. indep.?

Sol: $A\vec{x} = \vec{0}$. Aug. mat.
$$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix} \sim \underset{\text{REF}}{\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{bmatrix}}$$

⇒ x_1, x_2, x_3 are basic variables, no free vars.

⇒ no nontriv. solutions

⇒ columns of A are lin. indep.

Sets of one or two vectors.

- $\{\vec{v}\}$ set of 1 vector is lin. indep. iff $\vec{v} \neq \vec{0}$.

Ex: (a) $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -8 \\ -4 \end{bmatrix}$ (b) $\vec{v}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -8 \\ -4 \end{bmatrix}$ Q: are these two sets lin. indep.?

Sol: (a) notice: $\vec{v}_2 = -4\vec{v}_1$ a scalar multiple $\Rightarrow$ $4\vec{v}_1 + \vec{v}_2 = \vec{0}$ lin. dep. rel.

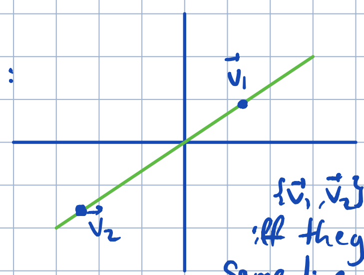
(b) $\vec{v}_1, \vec{v}_2$ are not multiples of one another. Assume $c\vec{v}_1 + d\vec{v}_2 = 0$ $(*)$ lin. dep. rel.

if $c \neq 0$, then $(*) \Rightarrow \vec{v}_1 = -\frac{d}{c}\vec{v}_2$ - impossible. Thus, $c=0$. Similarly, $d=0$.

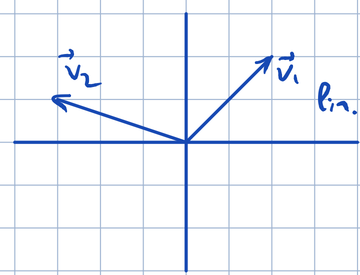
Hence, in $(*)$, $c=d=0$ $\Rightarrow$ $\{\vec{v}_1, \vec{v}_2\}$ lin. indep. set

- A set of two vectors $\{\vec{v}_1, \vec{v}_2\}$ is lin. dep. iff at least one of the vectors is a multiple of the other.

Geometrically:



$\{\vec{v}_1, \vec{v}_2\}$ lin. dep. iff they are on the same line through the origin



lin. indep.

Sets of ≥ 2 vectors

Theorem ("characterization of lin. dep. sets")

A set $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ of ≥ 2 vectors is lin. dep. iff at least one of the vectors in S is a lin. comb. of others.

In fact, if S is lin. dep. and $\vec{v}_1 \neq \vec{0}$, then some $\vec{v}_j$ (with $j>1$) is a lin. comb. of preceding vectors $\vec{v}_1, \ldots, \vec{v}_{j-1}$.

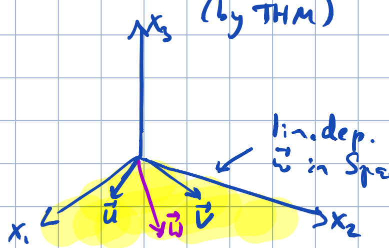
Warning: Thm doesn't say that every vector in S is a lin. comb. of others.

Ex: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ lin. dep. not a lin. comb.

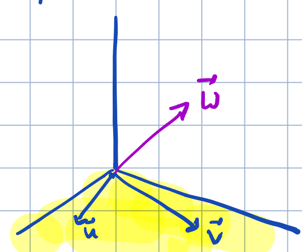
Ex: $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ $\vec{u}, \vec{v}$ lin. indep. (not multiples of each other)

Span$\{\vec{u}_1, \vec{u}_2\}$ = x_1, x_2 plane

$w \in$ Span$\{\vec{u}, \vec{v}\}$ iff $\{\vec{u}, \vec{v}, \vec{w}\}$ is a lin. dep. set (by THM)



lin. dep. $\vec{w}$ in Span$\{\vec{u}, \vec{v}\}$



lin. indep. $\vec{w}$ not in Span$\{\vec{u}, \vec{v}\}$

Theorem: If a set contains more vectors than there are entries in each vector, then the set is lin. dep. I.e. any set $\{\vec{v}_1, \dots, \vec{v}_p\}$ in $\mathbb{R}^n$ is lin. dep. if $\boxed{p > n}$.

Argument: $A = [\vec{v}_1 \dots \vec{v}_p]$ $\xrightarrow{A\vec{x}=\vec{0}:}$ n equations on p variables, # eqs < # vars ⇒ there are free variables

⇒ $A\vec{x} = 0$ admits a nontriv. sol. ⇒ columns of A are lin. dep. ☒

- If a set $S = \{\vec{v}_1, \dots, \vec{v}_p\}$ in $\mathbb{R}^n$ contains a zero vector, then S is lin. dep.
  (E.g. if $\vec{v}_1 = \vec{0}$. Then $1 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + \dots + 0 \cdot \vec{v}_p = \vec{0}$ lin. dep. rel.)

Ex: (a) $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}$

lin. dep. $(\underset{3}{\overset{}{p}} > \underset{2}{n})$, i.e. $p = 3$, $n = 2$

(b) $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}$

lin. dep.

(c) $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}3\\6\\9\\1\end{bmatrix}$

neither vector is a multiple of the other
⇒ lin. indep.