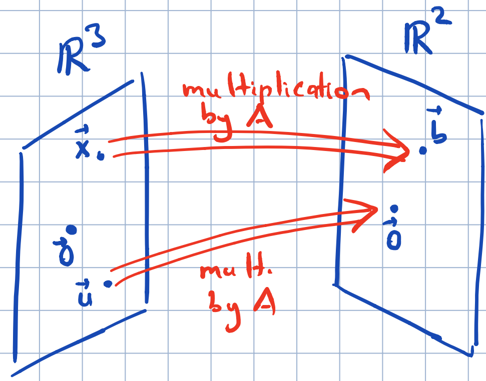
1/29/2020

1.8. Linear transformations

Ex: $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ A "acts" on vectors $\vec{x} \in \mathbb{R}^3$ by transforming $\vec{x}$ into $A\vec{x} \in \mathbb{R}^2$

$$\underbrace{A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 7 \\ 3 \end{bmatrix}}_{\vec{b}}, \quad A\underbrace{\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}}_{\vec{u}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\vec{0}}$$



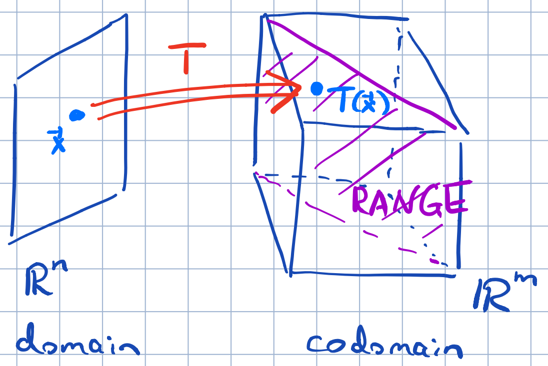
A transformation (or function, or mapping) T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule which assigns to each vector $\vec{x} \in \mathbb{R}^n$ a vector $T(\vec{x}) \in \mathbb{R}^m$

$\mathbb{R}^n$ - "domain" of T, $\mathbb{R}^m$ - "codomain"

Notation: $T: \underset{\text{domain}}{\mathbb{R}^n} \to \underset{\text{codomain}}{\mathbb{R}^m}$

for $x \in \mathbb{R}^n$, $T(\vec{x}) \in \mathbb{R}^m$ - "image of $\vec{x}$" (under the action of T)

Range of T = set of all images $T(\vec{x})$



- Given A an $m \times n$ matrix, we have a matrix transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ (# columns, # rows) with $T(\vec{x}) = A\vec{x}$. Notation for such transf.: $\vec{x} \mapsto A\vec{x}$

Ex: $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$ define $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\vec{x}) = A\vec{x}$

(a) find image $T(\vec{u})$ for $\vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Sol: $\begin{bmatrix} 1 \cdot 2 + (-3)(-1) \\ 3 \cdot 2 + 5 \cdot (-1) \\ (-1) \cdot 2 + 7\,(-1) \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$

(b) Solve $T(\vec{x}) = \underbrace{\begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}}_{\vec{b}}$ for $\vec{x}$. <u>Sol</u>: Aug. Mat of $A\vec{x}=\vec{b}$ $\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$ ②

$\Rightarrow$ $x_1 = -3/2$, $x_2 = 1/2$ $\Rightarrow$ $\vec{x} = \begin{bmatrix} -3/2 \\ 1/2 \end{bmatrix}$ – its image is $\vec{b}$!

(c) Is there more than one $\vec{x}$ s.t. $T(\vec{x}) = \vec{b}$?

<u>Sol</u>: $A\vec{x} = \vec{b}$ has a unique sol. (no free vars) $\Rightarrow$ NO, there is only one $\vec{x}$ whose image is $\vec{b}$.

(d) Is $\vec{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ in the range of T ?

<u>Sol</u>: $A\vec{x} = \vec{c}$ does sol. exist? Aug. Mat. $\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ \boxed{0 \quad 0 \quad -35} \end{bmatrix}$ $\Rightarrow$ system is inconsistent, solution does not exist

$\Rightarrow$ NO

<u>def</u> A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is <u>linear</u> if

(i) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v} \in \mathbb{R}^n$

(ii) $T(c\vec{u}) = c\,T(\vec{u})$ for all $c \in \mathbb{R}$, $\vec{u} \in \mathbb{R}^n$

<u>Main example</u>: every matrix transf. $T: \vec{x} \mapsto A\vec{x}$ is a linear transf.

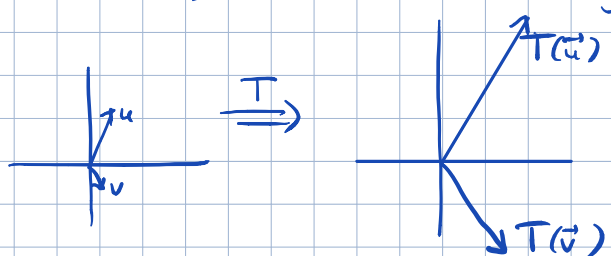
(Indeed: $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$, $A(c\vec{u}) = c(A\vec{u})$)

• Properties of a lin. transf. :
- $T(\vec{0}) = \vec{0}$
- $T(c\vec{u} + d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$

more generally: $T(c_1\vec{v}_1 + \ldots + c_p\vec{v}_p) = c_1T(\vec{v}_1) + \ldots + c_pT(\vec{v}_p)$ ← in Engineering – "superposition principle"

<u>Ex</u>: for r a scalar, define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\vec{x}) = r\vec{x}$. For $0 \le r < 1$, T is called "contraction"; for $r > 1$, T – "dilation"

E.g. $r = 3$



<u>T is linear</u>:

<u>check</u>: $T(c\vec{u} + d\vec{v}) = 3(c\vec{u} + d\vec{v})$
$= c(3\vec{u}) + d(3\vec{v})$
$= c(T\vec{u}) + d(T\vec{v})$

1.9 Matrix of a lin. transf.

Ex: $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ its columns: $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Suppose $T: \mathbb{R}^2 \to \mathbb{R}^3$ a lin. transf. s.t. $T(\vec{e}_1) = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$

Find $T(\vec{x})$ for arbitrary $\vec{x} \in \mathbb{R}^2$.

Sol: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{e}_1} + x_2 \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{e}_2} \Rightarrow T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2)$

$= x_1 T(\vec{e}_1) + x_2 T(e_2) =$

linearity

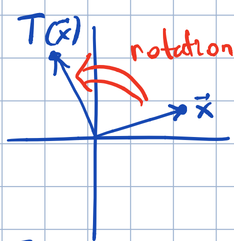
$$= x_1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}}_{T(\vec{e}_1)} + x_2 \underbrace{\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}}_{T(\vec{e}_2)} = \underbrace{\boxed{\begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 3 & 5 \end{bmatrix}}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\vec{x}$$

THM Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a lin. transf. Then there exists a unique matrix A s.t. $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

In fact, $\underbrace{A = [T(\vec{e}_1) \cdots T(\vec{e}_n)]}_{\text{stand. matrix of lin. transf. } T}$ where $\vec{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow$ j-th place is the j-th column of identity matrix I_n in $\mathbb{R}^n$.

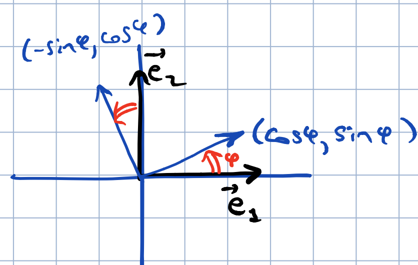
Ex: T - dilation, $r=3$ $\quad T: \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{e}_1} \mapsto \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{e}_2} \mapsto \begin{bmatrix} 0 \\ 3 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ - stand. matrix of T.

Ex: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$



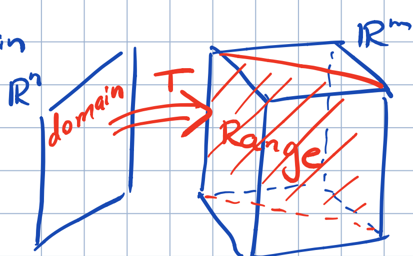
rotation by 90° counterclockwise

$A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$ - stand. mat. of rotation by angle φ

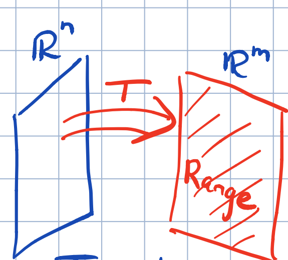


def A mapping $T: \mathbb{R}^n \to \mathbb{R}^m$ is onto $\mathbb{R}^m$ if each $\vec{b} \in \mathbb{R}^m$ the image of at least one $\vec{x} \in \mathbb{R}^n$

- T is onto iff range = codomain



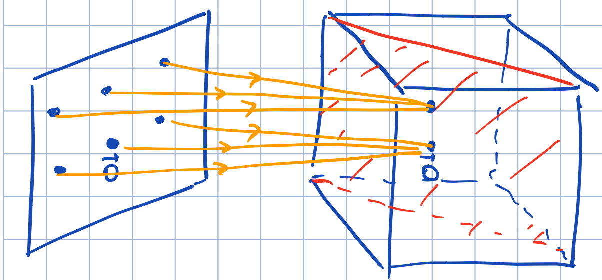
T not onto



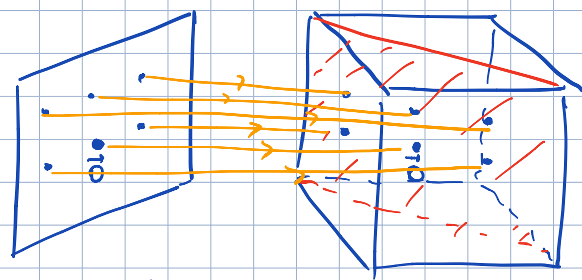
T onto $\mathbb{R}^m$

def $T: \mathbb{R}^n \to \mathbb{R}^m$ is <u>one-to-one</u> if each $\vec{b} \in \mathbb{R}^m$ is the image of <u>at most one</u> $\vec{x} \in \mathbb{R}^n$

- T is 1-1 iff $T(\vec{x}) = \vec{b}$ for each $\vec{b}$ has a unique sol., or none at all.



T not 1-1



T 1-1

Ex: T - lin. mapping with stand. mat. $A = \begin{bmatrix} 1 & 2 & -3 & 2 \\ 0 & 4 & 1 & -1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

Q: (a) is $T: \mathbb{R}^4 \to \mathbb{R}^3$ onto?
(b) is T 1-1?

Sol: (a) A has a pivot in every row $\Rightarrow$ $A\vec{x} = \vec{b}$ consistent $\forall \vec{b}$ $\Rightarrow$ T onto

(b) $A\vec{x} = \vec{b}$ has a free variable $\Rightarrow$ no uniqueness $\Rightarrow$ T not 1-1!

Rem: T is 1-1 iff eq. $T(\vec{x}) = \vec{0}$ has only the triv. sol.

THM: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a lin. transf., A - the stand. mat.

(a) T is <u>onto</u> iff columns of A span $\mathbb{R}^m$

(b) T is <u>1-1</u> iff columns of A are lin. indep.