1/31/2020

Rem: T is 1-1 iff eq. $T(\vec{x})=\vec{0}$ has only the triv. sol.

**THM:** Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a lin. transf., A - the stand. mat.

(a) T is onto iff columns of A span $\mathbb{R}^m$ [pivot in each row]

(b) T is 1-1 iff columns of A are lin. indep. [pivot in each column]

## 2.1 | Matrix operations

$$A=\begin{array}{c c}\left[\begin{array}{ccccc} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{array}\right] & \text{row } i \\ \begin{array}{ccccc} \vec{a}_1 & & \vec{a}_j & & \vec{a}_n \end{array} & \\ \text{column } j & \end{array}$$

$(i,j)$-entry of A

Zero matrix 0 (of size $m\times n$) - all entries zeros

- for A, B of same size $m\times n$, can form a sum $A+B$, $(A+B)_{ij}=a_{ij}+b_{ij}$

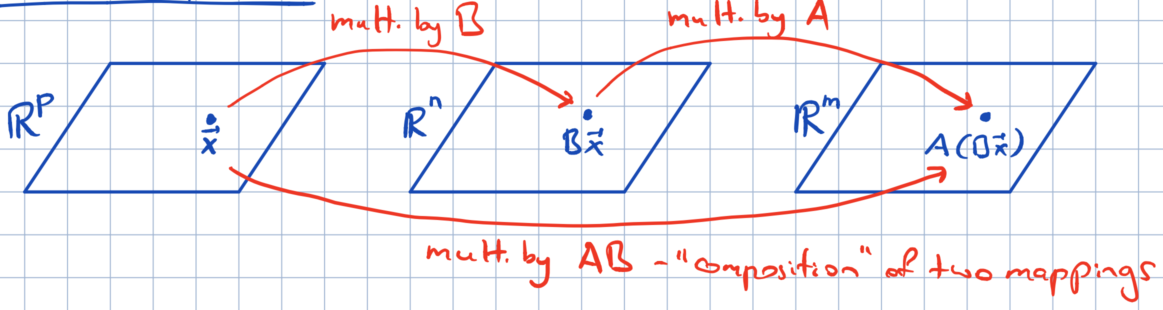
Ex: $A=\begin{bmatrix}1&3&0\\2&4&-1\end{bmatrix}$ $B=\begin{bmatrix}0&-1&3\\1&2&0\end{bmatrix}$ $A+B=\begin{bmatrix}1&2&3\\3&6&-1\end{bmatrix}$

- scalar multiples $\underset{\uparrow \atop \text{scalar}}{c}A$, $(cA)_{ij}=c\,a_{ij}$

Ex: $2B=\begin{bmatrix}0&-2&6\\2&4&0\end{bmatrix}$, $A+2B=\begin{bmatrix}1&1&6\\4&8&-1\end{bmatrix}$

properties - as for vector operations: $c(A+B)=cA+cB$ etc.

matrix multiplication



- want a matrix AB s.t. $A(B\vec{x})=(AB)\vec{x}$ for any $\vec{x}$.

$$A(B\vec{x})=A(x_1\vec{b}_1+\dots+x_p\vec{b}_p)=x_1Ab_1+\dots+x_pAb_p=$$

$$=\underbrace{[A\vec{b}_1 \cdots A\vec{b}_p]}_{AB}\vec{x}$$

**def** If $A$ is an $m\times n$ matrix, $B$ - $n\times p$ matrix, then the product $AB$ is an $m\times p$ matrix,

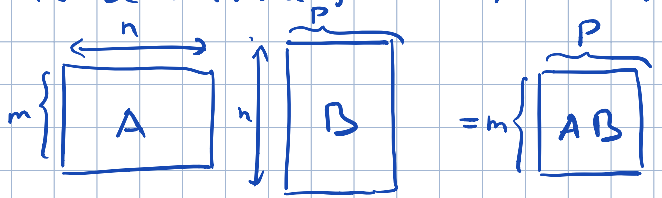
$$AB = A[\vec b_1 \cdots \vec b_p] = \boxed{[A\vec b_1 \cdots A\vec b_p]}$$

- multiplication of matrices corresponds to composition of lin. transf.

**Ex:** $A=\begin{bmatrix}1&3\\-2&-1\end{bmatrix}$ $\quad B=\left[\underset{\vec b_1}{\boxed{\begin{matrix}2\\0\end{matrix}}}\ \underset{\vec b_2}{\boxed{\begin{matrix}1\\3\end{matrix}}}\ \underset{\vec b_3}{\boxed{\begin{matrix}0\\5\end{matrix}}}\right]$

**Sol:** $A\vec b_1=\begin{bmatrix}2\\-4\end{bmatrix}$ $\quad A\vec b_2=\begin{bmatrix}10\\-5\end{bmatrix}$ $\quad A\vec b_3=\begin{bmatrix}15\\-5\end{bmatrix}$ $\quad \Rightarrow AB=\begin{bmatrix}2&10&15\\-4&-5&-5\end{bmatrix}$

- for $AB$ to be defined, need # columns (A) = # rows (B)



$(m\times n$ - matrix$)\cdot(n\times p$ - matrix$)$ = $m\times p$ matrix

- Row-column rule for $AB$:

$$(AB)_{ij} = a_{i1}b_{1j}+a_{i2}b_{2j}+\dots+a_{in}b_{nj}$$

Ex: $\begin{bmatrix}1&3\\-2&-1\end{bmatrix}\begin{bmatrix}2&1&0\\0&3&5\end{bmatrix}=\begin{bmatrix}*&*&1\cdot0+3\cdot5\\ *&*&*\end{bmatrix}$ ← entry (1,3)

$\begin{bmatrix}1&3\\-2&-1\end{bmatrix}\begin{bmatrix}2&1&0\\0&3&5\end{bmatrix}=\begin{bmatrix}*&*&*\\ *&(-2)\cdot1+(-1)\cdot3&*\end{bmatrix}$ ← entry (2,2)

**Properties:** $A(BC)=(AB)C$, $(A+B)C=AC+BC$, $\boxed{I_mA = A = AI_n}$

**WARNINGS** 1. Generally, $AB\neq BA$ ← Ex: $A=\begin{bmatrix}0&1\\0&0\end{bmatrix}$ $B=\begin{bmatrix}0&0\\1&0\end{bmatrix}$

$AB=\begin{bmatrix}1&0\\0&0\end{bmatrix}\neq BA=\begin{bmatrix}0&0\\0&1\end{bmatrix}$

2. Cancellation laws don't hold:

$$AC=BC \not\Rightarrow A=B$$

3. $AB=0 \not\Rightarrow A=0$ or $B=0$

- If $A$ an $n\times n$ matrix, $k\geq1$, $A^k=\underbrace{A\cdot A\cdots A}_{k}$ - $k$-th power of $A$

· **Transpose** for $A$ an $m\times n$ matrix, its transpose $A^T$ is an $n\times m$ matrix whose columns are formed from respective rows of $A$.

Ex: $A=\begin{bmatrix}a&b\\c&d\end{bmatrix} \Rightarrow A^T=\begin{bmatrix}a&c\\b&d\end{bmatrix}$, $B=\begin{bmatrix}1&4\\2&5\\3&6\end{bmatrix} \Rightarrow B^T=\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}$

Properties: $(A^T)^T=A$, $(A+B)^T=A^T+B^T$, $(cA)^T=cA^T$, $\boxed{(AB)^T=B^TA^T}$

reverse order !!

## 2.2 The inverse of a matrix

$A$ $n\times n$ matrix is invertible if there is an $n\times n$ mat. $C$ s.t. $CA=\underbrace{I}_{I_n}$ and $AC=I$

Then $C$ is called the inverse of $A$.

It is unique (if exists), notation: $A^{-1}$. Thus, $A^{-1}A=I$, $AA^{-1}=I$.

a non-invertible $A$ is called "singular"

Ex: $A=\begin{bmatrix}2&5\\-3&-7\end{bmatrix}$, $C=\begin{bmatrix}-7&-5\\3&2\end{bmatrix}$ $AC=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, $CA=\begin{bmatrix}1&0\\0&1\end{bmatrix}$, thus $A^{-1}=C$.

Thm: Let $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$. If $ad-bc\neq 0$, then $A$ is invertible and

$$A^{-1}=\frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}.$$

If $\boxed{ad-bc}=0$, then $A$ is non-invertible

"determinant" det $A$

Ex: $A=\begin{bmatrix}2&5\\-3&-7\end{bmatrix}$, $\det A=2(-7)-5(-3)=1$, $A^{-1}=\begin{bmatrix}-7&-5\\3&2\end{bmatrix}$, cf. prev. ex.

Ex: $\begin{bmatrix}1&2\\3&4\end{bmatrix}^{-1}=\frac{1}{-2}\begin{bmatrix}4&-2\\-3&1\end{bmatrix}=\begin{bmatrix}-2&1\\3/2&-1/2\end{bmatrix}$

· If $A$ is an invertible $n\times n$ matrix, then for each $\vec{b}\in\mathbb{R}^n$, eq. $A\vec{x}=\vec{b}$ has unique sol. $\boxed{\vec{x}=A^{-1}\vec{b}.}$

· Properties: $(A^{-1})^{-1}=A$, $(AB)^{-1}=\underbrace{B^{-1}A^{-1}}_{\text{reverse order}}$, $(A^T)^{-1}=(A^{-1})^T$

## Algorithm for finding $A^{-1}$

Row reduce the "augmented matrix" $[A \,\vdots\, I]$ - $n \times 2n$ matrix

if $A$ is invertible, RREF is: $[I \,\vdots\, A^{-1}]$.

<u>Ex:</u> $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$[A \,\vdots\, I] = \left[\begin{matrix}\underbrace{\begin{matrix}1 & 2\\ 3 & 4\end{matrix}}_{A} & \vdots & \underbrace{\begin{matrix}1 & 0\\ 0 & 1\end{matrix}}_{I}\end{matrix}\right] \sim \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3/2 & -1/2 \end{array}\right]$$

$$\sim \left[\begin{matrix}\underbrace{\begin{matrix}1 & 0\\ 0 & 1\end{matrix}}_{I} & \vdots & \underbrace{\begin{matrix}-2 & 1\\ 3/2 & -1/2\end{matrix}}_{A^{-1}}\end{matrix}\right]$$