2/14/2020

3.3 Cramer's rule. Volume and linear transformations.

A $n\times n$ matrix, $\vec{b}\in\mathbb{R}^n$. Let $A_i(\vec{b}) = [\vec{a}_1 \cdots \underset{\uparrow \text{ col } i}{\vec{b}} \cdots \vec{a}_n]$

Thm (Cramer's rule)

for A invertible $n\times n$, $\vec{b}\in\mathbb{R}^n$, the unique solution of $A\vec{x}=\vec{b}$ has entries

$$\boxed{x_i = \frac{\det A_i(\vec{b})}{\det A}}, \quad i=1,\dots,n$$

Ex:

$$\begin{aligned} 4x_1 + 5x_2 &= 2 \\ 2x_1 + 3x_2 &= 6 \end{aligned}$$

$\uparrow$ $\vec{b}$

Solve using Cramer's rule:

$A = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$ det $= 2$

$A_1(\vec{b}) = \begin{bmatrix} 2 & 5 \\ 6 & 3 \end{bmatrix}$ det $= -24$

$A_2(\vec{b}) = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}$ det $= 20$

$$\boxed{\begin{aligned} x_1 &= \frac{-24}{2} = -12 \\ x_2 &= \frac{20}{2} = 10 \end{aligned}}$$

Ex: for which s (parameter), system

$$\begin{aligned} 3s x_1 - 2x_2 &= 1 \\ -6x_1 + s x_2 &= 2 \end{aligned}$$

(a) has a unique solution?

(b) write the sol. using Cramer's rule

Sol: $A = \begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix}$

det $= 3s^2 - 12 = 3(s-2)(s+2)$

(a): det $\neq 0$ iff $s \neq \pm 2$

$A_1(\vec{b}) = \begin{bmatrix} 1 & -2 \\ 2 & s \end{bmatrix}$

det $= s+4$

$A_2(\vec{b}) = \begin{bmatrix} 3s & 1 \\ -6 & 2 \end{bmatrix}$

det $= 6s+6 = 6(s+1)$

(b): $x_1 = \dfrac{s+4}{3(s-2)(s+2)}$

$x_2 = \dfrac{6(s+1)}{3(s-2)(s+2)} = 2\dfrac{(s+1)}{(s-2)(s+2)}$

Formula for A^{-1}

for A invertible $n\times n$ matrix,

$$A^{-1} = \frac{1}{\det A}\underbrace{\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}}_{\text{"adjugate" of } A, \ \operatorname{adj} A}$$

or equivalently, $(A^{-1})_{ij} = \dfrac{C_{ji}}{\det A}$

Ex: $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \\ 2 & 1 & -6 \end{bmatrix}$ find $(A^{-1})_{12}$

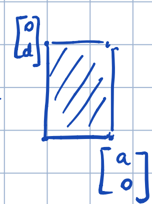
Sol: $\det A = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1$ $\qquad C_{21} = -\begin{vmatrix} 1 & 1 \\ 1 & -6 \end{vmatrix} = 7$

$\Rightarrow (A^{-1})_{12} = \dfrac{C_{21}}{\det A} = 7$

Determinants as area or volume

Thm (a) If $A = [\vec{a}_1, \vec{a}_2]$ is a 2×2 matrix, the area of the parallelogram determined by $\vec{a}_1, \vec{a}_2$ is $|\det A|$

(b) If $A = [\vec{a}_1\ \vec{a}_2\ \vec{a}_3]$ is a 3×3 matrix, the volume of the parallelipiped determined by $\vec{a}_1, \vec{a}_2, \vec{a}_3$ is $|\det A|$

Ex: $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ Area of 

$= |ad| = |\det A|$

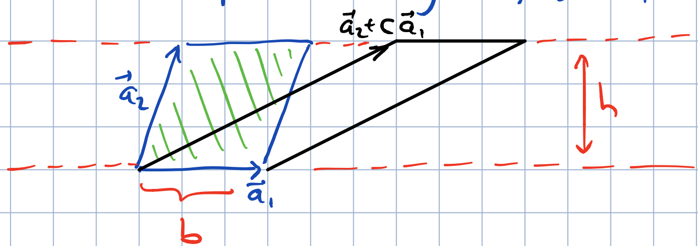
Idea of proof of (a):

$A \underset{(i)}{\sim}$ diagonal matrix $\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$

$\left.\begin{array}{l} \text{(i) col. replacements} \\ \text{(ii) col. interchanges} \end{array}\right\}$ change neither $|\det A|$, nor Area

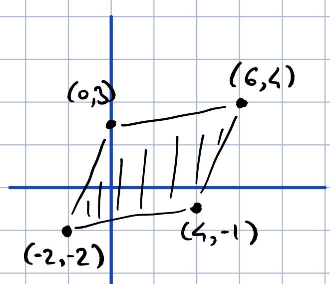
(ii) Area (parall. det. by $\vec{a}_2, \vec{a}_1$) = Area (parall. det. by $\vec{a}_1, \vec{a}_2$)

(i) Area (parall. det. by $\vec{a}_1, \vec{a}_2 + c\vec{a}_1$) = Area (parall. det. by $\vec{a}_1, \vec{a}_2$)

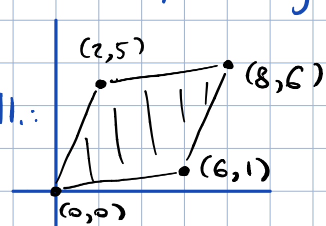


both areas $= \begin{pmatrix} \text{base} \\ b \end{pmatrix} \cdot \begin{pmatrix} \text{height} \\ h \end{pmatrix}$

Ex: find the area of the parallelogram with vertices at $(-2,-2)$, $(0,3)$, $(4,-1)$, $(6,4)$



Sol: translate the parall. by $(2,2)$, to have $\vec{0}$ as a vertex

new parall.: 

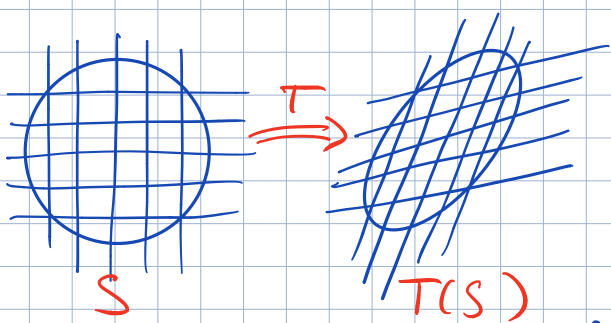
$\text{Area} = \left| \det \begin{bmatrix} 2 & 6 \\ 5 & 1 \end{bmatrix} \right| = |-28| = 28$

How areas / volumes are changed by a linear transformation?

THM* (a) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a lin. transf. determined by a 2×2 matrix A.
If S is a parallelogram in $\mathbb{R}^2$, then $(\text{Area of } T(S)) = |\det A| \cdot (\text{Area of } S)$

(b) If $T: \mathbb{R}^3 \to \mathbb{R}^3$ is determined by a 3×3 matrix A and S - parallelipiped in $\mathbb{R}^3$,
then $(\text{Volume of } T(S)) = |\det A| \, (\text{Volume of } S)$.

THM* generalizes to finite area regions S of $\mathbb{R}^2$ / finite volume regions of $\mathbb{R}^3$



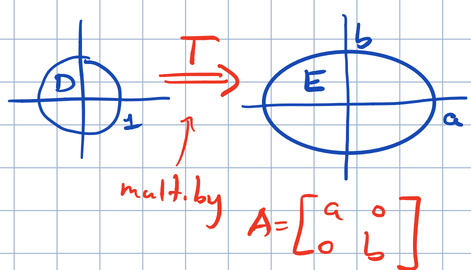
$\text{Area}(T(S)) = |\det A| \cdot \text{Area}(S)$

S can be approximated by a union of little squares

$T(S)$ - union of little parallelograms $= T(\text{little squares})$

Ex: Let E be a region in $\mathbb{R}^2$ bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. $\text{Area}(E) = ?$

Sol:



Indeed: $T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{array}{l} u_1 = \frac{x_1}{a} \\ u_2 = \frac{x_2}{b} \end{array}$

$\vec{u}$ is the unit disk D: $u_1^2 + u_2^2 \le 1$ iff $\vec{x}$ in E: $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \le 1$

Thus: $\text{Area}(E) = \underbrace{|\det A|}_{ab} \; \underbrace{\text{Area}(D)}_{\pi \cdot 1^2} = \pi a b$.