LAST TIME

THM (spanning set theorem)

Let $S=\{\vec{v}_1,\dots,\vec{v}_p\}$ be a spanning set for H.

(a) if a vector $\vec{v}_k$ from S is a lin. comb. of other vectors in S, then the set formed from S by deleting $\vec{v}_k$ still spans H.

(b) if $H\neq\{\vec{0}\}$, some subset of S is a basis for H.

A basis for H is: - the smallest spanning set for H ← can delete lin.dep. vectors from a spanning set for H. When we arrive to a lin. indep. subset, if we delete one more, the result will no longer span H!

- the largest lin. indep. set in H.

Ex: $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\end{bmatrix}\right\} \xrightarrow[\text{to a basis}]{\text{enlarge}} \left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix}\right\} \xrightarrow[\text{enlargement}]{\text{further}} \left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix},\begin{bmatrix}7\\8\\9\end{bmatrix}\right\}$

further shrinking | shrink to a basis

lin. indep. set, does not span $\mathbb{R}^3$ | basis in $\mathbb{R}^3$ | spans $\mathbb{R}^3$, lin. dep.

4.4. Coordinate systems

a basis $\mathcal{B}$ (with n vectors) for V imposes a "coord. system" on V which makes V "act like $\mathbb{R}^n$"

Thm (Unique representation theorem)

Let $\mathcal{B}=\{\vec{b}_1,\dots,\vec{b}_n\}$ be a basis for a v.sp. V. Then for each $\vec{x}\in V$ there exists a unique set of scalars $c_1,\dots,c_n$ s.t. $\vec{x}=c_1\vec{b}_1+\dots+c_n\vec{b}_n$ $(*)$

↑ from spanning property of $\mathcal{B}$ | ↑ from lin. indep. property

def weights $c_1,\dots,c_n$ in $(*)$ - coordinates of $\vec{x}$ rel. to $\mathcal{B}$ ($\mathcal{B}$-coordinates of $\vec{x}$)

$[\vec{x}]_{\mathcal{B}}=\begin{bmatrix}c_1\\ \vdots\\ c_n\end{bmatrix}\in\mathbb{R}^n$ - coord. vector of $\vec{x}$ (rel. to $\mathcal{B}$) / $\mathcal{B}$-coord. vector of $\vec{x}$

Mapping $\begin{matrix}V\to\mathbb{R}^n\\ \vec{x}\mapsto[\vec{x}]_{\mathcal{B}}\end{matrix}$ - coord. mapping defined by $\mathcal{B}$.

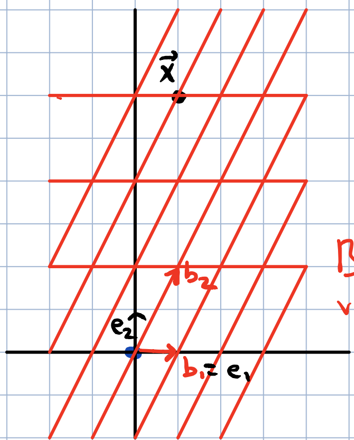
Ex: $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ basis in $\mathbb{R}^2$, with $\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Q: $\vec{x} \in \mathbb{R}^2$ with $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ find $\vec{x}$.

Sol: $\vec{x} = -2\vec{b}_1 + 3\vec{b}_2 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$.

Note: $\vec{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix} = 1\vec{e}_1 + 6\vec{e}_2$

thus, coord. vector of $\vec{x}$ rel. to stand. basis $\mathcal{E} = \{\vec{e}_1, \vec{e}_2\}$ is $[\vec{x}]_{\mathcal{E}} = \vec{x}$.



$\mathcal{B}$-coord grid vs. stand. coord. grid on $\mathbb{R}^2$

Coordinates in $\mathbb{R}^n$

Ex: $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ in $\mathbb{R}^2$, with $\vec{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$. **Q:** find $[\vec{x}]_{\mathcal{B}}$

Sol: $c_1\vec{b}_1 + c_2\vec{b}_2 = \vec{x} \iff \underbrace{\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}}_{P_{\mathcal{B}}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, where the columns of $P_{\mathcal{B}}$ are $\vec{b}_1, \vec{b}_2$ and the right-hand side is $\vec{x}$.

$P_{\mathcal{B}}$ - matrix changing $\mathcal{B}$-coords of $\vec{x}$ to stand. coords

Sol: $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

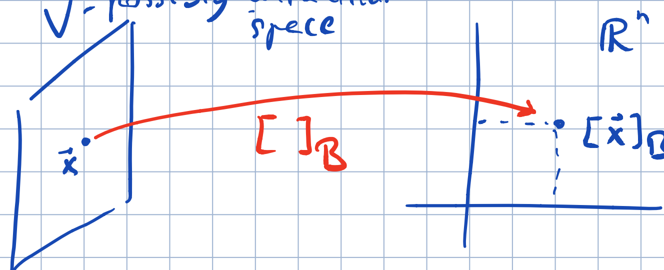
- for any basis $\mathcal{B}$ in $\mathbb{R}^n$, let $P_{\mathcal{B}} = [\vec{b}_1 \cdots \vec{b}_n]$

then $\vec{x} = c_1\vec{b}_1 + \dots + c_n\vec{b}_n \iff \boxed{\vec{x} = \underbrace{P_{\mathcal{B}}}_{\text{change-of-coord. mat. from } \mathcal{B} \text{ to } \mathcal{E}} [\vec{x}]_{\mathcal{B}}} \implies \underbrace{P_{\mathcal{B}}^{-1}}_{\text{mat. of coord. mapping}} \vec{x} = [\vec{x}]_{\mathcal{B}}$

mat. of coord. mapping $\vec{x} \to [\vec{x}]_{\mathcal{B}}$, 1-1, onto

Coord. mapping

V - possibly unfamiliar space



$[\vec{x}]_{\mathcal{B}}$ - "new name for $\vec{x}$"

Thm Let $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ be a basis for V. Then the coord mapping $\vec{x} \longmapsto [\vec{x}]_{\mathcal{B}}$ is a <u>1-1</u> lin. transf. from V <u>onto</u> $\mathbb{R}^n$.

- In particular, $[c_1\vec{u}_1+\dots+c_p\vec{u}_p]_{\mathcal{B}} = c_1[\vec{u}_1]_{\mathcal{B}}+\dots+c_p[\vec{u}_p]_{\mathcal{B}}$ - preserves lin. comb.
- A lin. mapping $T: V \to W$ which is 1-1 and onto is called an isomorphism.
  - every vector space calculation in V is reproduced in W and vice versa.
  So, V and W are "same".
- A v.sp. V with a basis of n vectors is indistinguishable from $\mathbb{R}^n$.

Ex: $\mathcal{B}=\{1,t,t^2,t^3\}$ stand. basis in $\mathbb{P}_3$

p in $\mathbb{P}_3$ is $p(t)= a_0+a_1t+a_2t^2+a_3t^3$ - lin. comb. of stand. basis vectors

$$[p]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

coord. mapping

$$p \mapsto [p]_{\mathcal{B}}$$
$$\mathbb{P}_3 \to \mathbb{R}^4$$

$\mathbb{P}_3$ "acts" like $\mathbb{R}^4$

Ex: $p_1 = 1+2t^2$
$p_2 = 4+t+5t^2$
$p_3 = 3+2t$

check that $\{p_1,p_2,p_3\}$ are lin. dep. in $\mathbb{P}_2$ using coord. vectors

Sol: Aug. mat. of $A\vec{x}=0$

$$\begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 5 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{col. of } A \text{ are lin. dep.}$$

$[p_1]_{\mathcal{B}}\ [p_2]_{\mathcal{B}}\ [p_3]_{\mathcal{B}}$

$$\sim \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 - 5x_3 = 0$
$x_2 + 2x_3 = 0$ $\Rightarrow$

$\Rightarrow 5[p_1]_{\mathcal{B}} - 2[p_2]_{\mathcal{B}} + [p_3]_{\mathcal{B}} = 0$

$\Rightarrow 5p_1 - 2p_2 + p_3 = 0$ relation for polynomials

Ex: $\vec{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$ $\mathcal{B}=\{\vec{v}_1,\vec{v}_2\}$ - basis for $H=\text{Span}\{\vec{v}_1,\vec{v}_2\}$

(a) is $\vec{x}$ in H (b) if yes, find $[\vec{x}]_{\mathcal{B}}$

Sol: $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{x}$ Aug. Mat. $\begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ $\Rightarrow$ sol. exists (a) - YES

$c_1=2, c_2=3$

(b) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Practice questions

① $\mathcal{B} = \{p_1 = 1+2t^2,\ p_2 = t,\ p_3 = 2+3t^2\}$ in $\mathbb{P}_2$ $\qquad p = 1+t+t^2$

Q: find $[p]_{\mathcal{B}}$

Sol: $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$c_1[p_1]_{\mathcal{E}} + c_2[p_2]_{\mathcal{E}} + c_3[p_3]_{\mathcal{E}} = p$ $\qquad \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ $\qquad \Rightarrow [p]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$-p_1 + p_2 + p_3 = p$

② $\mathcal{B} = \left\{ \underset{\vec{b}_1}{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}, \underset{\vec{b}_2}{\begin{bmatrix} 2 \\ 5 \end{bmatrix}} \right\}$ in $\mathbb{R}^2$

(a) find $P_{\mathcal{B}}$ $\quad \rightarrow \quad P_{\mathcal{B}} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

(b) $P_{\mathcal{B}}^{-1} = ?$ $\quad \rightarrow \quad P_{\mathcal{B}}^{-1} = -\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$

(c) find $\mathcal{B}$-coord. vector of $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ using (b)

$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$